STATISTICS 906

MIDTERM TEST OCTOBER 29, 2003

Name__________________

1. Suppose you have generated independent $U[0,1]$ random variables $U_1, U_2, \ldots$. Give a formula or algorithm for generating a single random variable X where X has:

a. the following probability density function:

$$f(x) = \frac{1}{2x\sqrt{2\pi}} \exp\{-\frac{(\log x - 2)^2}{8}\}.$$

b. probability density function

$$f(x) = \frac{1}{96} x^3 e^{-x/2}$$

for $0 \le x$, otherwise $f(x) = 0$.

c. A discrete random variable with probability function

$$P[X = x] = \frac{1}{kx}(.9)^x, x = 1, 2, \ldots, \text{ where } k = 2.3025.$$

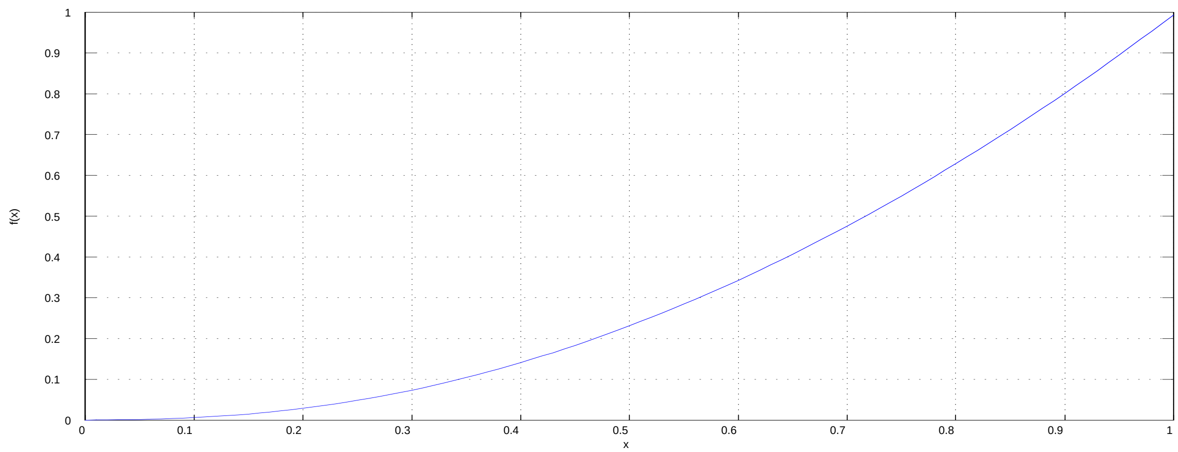
How many uniform random numbers are required on average to generate a single random variable with this distribution?

d. What is the probability density function of the random variables Z generated by the following Matlab code?

```
U=rand(1,100000);
V=rand(1,100000);
X=U(V<1-U);
W=rand(1,length(X));
Z=min(W.^(1/2),X);
```

2. Define the function whose graph appears in Figure 1;

$$f(x) = \frac{x^2}{1+e^{-5x}}$$



and suppose we wish to estimate the integral

$$= \int_0^1 f(x) 2x dx$$

using Monte Carlo integration. Give estimators based on a sequence of independent uniform[0,1] random variables $U_1, \ldots, U_n$ of this integral using the following methods. For each indicate how to assess their relative efficiency.

a. Importance sampling:

b. A control variate:

c. A stratified random sample with two strata, [0,0.75] and [0.75,1].

d. Guess the optimal sample sizes if we use the stratified sample with strata [0,.75] and [0.75,1]. You may pretend that your function is equal to the one you used as a contol variate in part (b).

3. A model for a financial time series S_t is written with stochastic differential equation in the form

$$dS_t = rS_t dt + \quad S_t^{0.5} dW_t$$

for a Wiener (standard Brownian motion) process W_t.

a. Give two methods for simulating S_2 starting with $S_0 = 10, r = 0.05, \quad = 0.2$ and step size 1.

b. Suppose independent simulations are conducted at two points in order to estimate the rho of a derivative which has payoff at maturity $T = 2$ given by

$$V(S_T) = (S_T - 10) \text{ if } 8 \le S_T \le 12$$

otherwise $V(S_T) = 0$.

Is it possible to replicate this derivative using the stock, a risk-free account, ordinary European call options and digital options (having payoff equal to \$1 if $S_T \ge K$) possibly with different strike prices?

c. We simulated the value of the discounted payoff $e^{-2r}V(S_T)$ under two different circumstances:

Method 1 using interest rate $r = 0.045$ and $r = 0.055$ and independent simulations, $n = 100,000$ at each of $r = 0.045$ and $r = 0.055$. .

Method 2 using interest rate $r = 0.045$ and $r = 0.055$ and common random numbers. The results of (ii) are as follows:

Number of simulations	r	estimate of $var(e^{-2r}V(S_T))$	average$(e^{-2r}V(S_T))$
$100,000$	0.045	0.46	0.57
$100,000$	0.055	0.43	0.63

The correlation coefficient between the vector of estimators at the two different values of r using Method 2 was 0.8. Use this data to estimate rho and estimate the efficiency of the use of common random numbers relative to the use of independent simulations as in Method 1.