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Stochastic Processes and their Applications 119 (2009) 655–675

stochastic processes and their applications

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Approximation of the tail probability of randomly weighted sums and applications

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Received 19 June 2007; received in revised form 3 March 2008; accepted 20 March 2008

Available online 24 March 2008

Abstract

Consider the problem of approximating the tail probability of randomly weighted sums $\sum_{i=1}^{n} \Theta_i X_i$ and their maxima, where $\{X_i, i \geq 1\}$ is a sequence of identically distributed but not necessarily independent random variables from the extended regular variation class, and $\{\Theta_i, i \geq 1\}$ is a sequence of nonnegative random variables, independent of $\{X_i, i \geq 1\}$ and satisfying certain moment conditions. Under the assumption that $\{X_i, i \geq 1\}$ has no bivariate upper tail dependence along with some other mild conditions, this paper establishes the following asymptotic relations:

$$\Pr\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}\Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{n}\Theta_i X_i > x\right) \sim \sum_{i=1}^{n}\Pr(\Theta_i X_i > x),$$

and

$$\Pr\left(\max_{1\leq k<\infty}\sum_{i=1}^{k}\Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{\infty}\Theta_i X_i^{+} > x\right) \sim \sum_{i=1}^{\infty}\Pr(\Theta_i X_i > x),$$

as $x \to \infty$. In doing so, no assumption is made on the dependence structure of the sequence $\{\Theta_i, i \geq 1\}$.

MSC: 60E20; 62P05; 91B30

Keywords: Randomly weighted sums; Asymptotics; Regular variation; Upper tail dependence; Ruin probability; Stochastic difference equations

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2008.03.004

1. Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of identically distributed but not necessarily independent random variables with a generic random variable (r.v.) X and common cumulative distribution (c.d.f.) F, and $\{\Theta_i, i \geq 1\}$ be another sequence of nonnegative r.v.'s, that is independent of the sequence $\{X_i, i \geq 1\}$. Note that $\{\Theta_i, i \geq 1\}$ are also generally dependent. In this paper, we discuss the tail probabilities of the randomly weighted sums

$$S_n := \sum_{i=1}^{n} \Theta_i X_i, \quad n \geq 1, \tag{1.1}$$

and their maxima $M_n := \max_{1\leq k\leq n} S_k$, $n \geq 1$.

The randomly weighted sums S_n and their maxima M_n are frequently encountered in various areas, especially in actuarial and economic situations. For example, in the actuarial context, the r.v.'s $\{X_i, i \geq 1\}$ are often interpreted as the liability risks while the weights $\{\Theta_i, i \geq 1\}$ stand for the financial risks, such as the discount factors. More specifically, if we regard X_i as the net loss, i.e. the total amount of premium incomes minus the total amount of claims for an insurance company during period i, then the sum S_n is the discounted losses accumulated from time 0 to time n. See Section 4.1 for detailed interpretation.

In this paper, we will focus on the case when the sequence $\{X_i, i \geq 1\}$ are heavy-tail distributed (i.e. their moment generating functions does not exist). Specifically, we suppose that $\{X_i, i \geq 1\}$ are from the extended regular variation class and have no bivariate upper tail dependence (see Definition 2.2 and Remark 2.1 for its accurate definition), and that the weights $\{\Theta_i, i \geq 1\}$ satisfy certain moment conditions. See Definition 2.1 for the concept of the extended regular variation class along with some other classes of heavy-tailed distributions. We will establish the following asymptotics in the present paper.

$$\Pr\left(\max_{1\leq k\leq n} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) \sim \sum_{i=1}^{n} \Pr(\Theta_i X_i > x), \tag{1.2}$$

and

$$\Pr\left(\max_{1\leq k<\infty} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i^+ > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x), \tag{1.3}$$

as $x \to \infty$ (see Theorem 3.1), where $x^+ = \max\{0, x\}$ for any real number x, and $a(x) \sim b(x)$ means that both $\limsup_{x\to\infty} \frac{a(x)}{b(x)} \leq 1$ and $\liminf_{x\to\infty} \frac{a(x)}{b(x)} \geq 1$ hold for two positive functions $a(x)$ and $b(x)$.

Due to the important role of the heavy-tailed distributions in many applied fields, there is a proliferate of research being conducted to estimating the tail probabilities of sums S_n and their maxima M_n with heavy-tail distributed r.v.'s $\{X_i, i \geq 1\}$. To our knowledge, however, all the results obtained so far are fully in line with ours, yet under different assumptions on $\{X_i, i \geq 1\}$ and/or $\{\Theta_i, i \geq 1\}$. All the existing literature assume $\{X_i, i \geq 1\}$ to be independent, and some of them even require the weights $\{\Theta_i, i \geq 1\}$ to be specially structured such that $\Theta_i = \prod_{k=1}^{i} Y_k$, $i \geq 1$, for an independent sequence of r.v.'s $\{Y_k, k \geq 1\}$.

Next, we shall have a review on the related literature. First look at those papers that consider the case when $\{X_i, i \geq 1\}$ are from the regular variation class $\mathcal{R}_{-\alpha}$ for $\alpha > 0$. Resnick and Willehens [23] established the following asymptotic result (1.4) under the assumption that $\{X_i, i \geq 1\}$ are nonnegative r.v.'s.

$$\Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i > x\right) \sim F(x) \sum_{i=1}^{\infty} \mathsf{E}[\Theta_i^{\alpha}]. \tag{1.4}$$

Later on, Goovaerts et al. [14] generalized their results by allowing $\{X_i, i \geq 1\}$ to be real-valued, and establishing the more general results that are similar to (1.2) and (1.3); also see [24]. The conditions imposed on $\{\Theta_i, i \geq 1\}$ by Resnick and Willehens [23], Goovaerts et al. [14], and Tang [24] for their results are all essentially the same as that for ours in the present paper (see assumptions **H1** and **H2** in Section 2). Under a different condition that is defined through the right endpoints of $\{\Theta_i, i \geq 1\}$, Chen et al. [6] achieved an asymptotic formula for the maxima M_n.

Besides the regular variation class, asymptotic results like (1.2) and (1.3) have also been attempted by several papers for $\{X_i, i \geq 1\}$ from some other wider class of heavy-tailed distributions. Tang and Tsitsiashvili [25] established the asymptotic formulation regarding the maxima M_n as in (1.2) for the case when $\{X_i, i \geq 1\}$ belong to the intersection of the long-tailed class and dominant variation class, and $\{\Theta_i, i \geq 1\}$ are structured through the independent sequence $\{Y_i, i \geq 1\}$ of r.v.'s as mentioned before. Later on, Wang and Tang [28] extended their results by allowing $\{\Theta_i, i \geq 1\}$ to be generally structured, and establishing the asymptotic results not only for the maxima M_n but also for the sums S_n. Papers considering the subexponential class, which is a very general class of heavy-tailed distributions, include [26,7]. While their results are applied to more general $\{X_i, i \geq 1\}$, more strong conditions are demanded on the weights $\{\Theta_i, i \geq 1\}$. Tang and Tsitsiashvili [26] suppose the weights to be bounded, while Chen and Su [7] assume some conditions on the density of Θ_i for each $i \geq 1$.

Note that the results in (1.2) hold for a fixed integer n. Theoretically, it is also interesting to investigate when they are valid uniformly for $n \geq 1$. The existing literature addressing this issue are [29,28], with the former discussing the maxima M_n for $\{X_i, i \geq 1\}$ from the consistent variation class, and the latter on both the sums S_n and the maxima M_n for $\{X_i, i \geq 1\}$ within the extended regular variation class. Moreover, another two interesting papers related to ours are [20,21], where the large deviation techniques are employed to obtain certain estimate of the tail probabilities of the maxima. It is also worth mentioning that many papers reviewed above were tackling the problem in the context of ruin theory, where the ruin probabilities are defined by the tail probabilities of the maximum in a discrete time risk model (see Section 4.1).

The rest of this paper is constructed as follows. Section 2 is the preliminary, where we recall some definitions and present some lemmas that are crucial to the proof of our main results. Section 3 states the main results along with some remarks. Section 4 is some applications of our main results. The proof of our main results and some lemmas are presented in Appendix.

2. Preliminary

Here and henceforth, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(x), b(x)$, throughout this paper, we write $a(x) \lesssim b(x)$ if $\limsup_{x\to\infty} \frac{a(x)}{b(x)} \leq 1$, $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} \frac{a(x)}{b(x)} \geq 1$, $a(x) \sim b(x)$ if both, and $a(x) \asymp b(x)$ if $0 < \liminf_{x\to\infty} \frac{a(x)}{b(x)} \leq \limsup_{x\to\infty} \frac{a(x)}{b(x)} < \infty$. We shall use the symbols $x^+ = \max\{x, 0\}$ for a real number x and $\overline{F} = 1 - F$ for a c.d.f F.

A r.v. X or its c.d.f. $F(x)$ satisfying $\overline{F}(x) > 0$ for all $x \in (-\infty, \infty)$ is heavy-tailed to the right, or simply heavy-tailed, if $\mathsf{E}e^{\gamma X} = \infty$ for all $\gamma > 0$. We recall here two important classes of heavy-tailed distributions as follows.

Definition 2.1. (1) Regular Variation ($\mathcal{R}_{-\alpha}$) Class. A r.v. X or its c.d.f. F on $(-\infty, +\infty)$ belongs to $\mathcal{R}_{-\alpha}$ for some $\alpha > 0$, if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \tag{2.1}$$

holds for any $y > 0$.

(2) Extended Regular Variation (*ERV*) Class. A r.v. X or its c.d.f. $F(x)$ on $(-\infty, +\infty)$ belongs to $ERV(-\alpha, -\beta)$ for $0 < \alpha \le \beta < \infty$ such that

$$s^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le s^{-\alpha}, \quad \text{for all } s \ge 1, \tag{2.2}$$

or equivalently

$$s^{-\alpha} \le \liminf_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le s^{-\beta}, \quad \text{for all } 0 < s \le 1. \tag{2.3}$$

In addition to the above two classes, there are some other important classes of heavy-tailed distributions known as the Consistent variation class $\mathcal{C}$, the Dominant variation class $\mathcal{D}$, the Subexponential class $\mathcal{S}$, and the Long-tailed class $\mathcal{L}$. These classes satisfy the following inclusion relations:

$$\mathcal{R}_{-\alpha} \subset ERV(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}. \tag{2.4}$$

Moreover, if $F \in ERV(-\alpha, -\beta)$, then it follows from Proposition 2.2.1 of [4] or Section 3.3 in [25] that, for any α' and β' satisfying $\alpha' < \alpha$ and $\beta' > \beta$, there exist positive constants C_i and D_i, $i = 1, 2$, such that

$$\frac{\overline{F}(y)}{\overline{F}(x)} \ge C_1 \left(\frac{x}{y}\right)^{\alpha'} \tag{2.5}$$

for all $x \ge y \ge D_1$, and

$$\frac{\overline{F}(y)}{\overline{F}(x)} \le C_2 \left(\frac{x}{y}\right)^{\beta'} \tag{2.6}$$

for all $x \ge y \ge D_2$. Due to the arbitrariness of $\beta' > \beta$, fixing the variable y in (2.6) immediately leads to

$$x^{-\beta^*} = o(\overline{F}(x)), \quad \text{for all } \beta^* > \beta. \tag{2.7}$$

For a comprehensive review on heavy-tailed distributions and their applications in insurance and finance, we refer to [4], [8], and [12].

Next, we recall the definition of the index of upper tail dependence for a bivariate random vector $\mathbf{X} = (X_1, X_2)$. By Sklar's Theorem (see [19] or [17]), we see that if $\mathbf{X}$ has continuous marginal distributions F_1 and F_2, then the dependence structure of X_1 and X_2 is completely determined by a bivariate copula function $C(u, v)$ and the joint distribution function of $\mathbf{X}$ is given by $C(F_1(x), F_2(x))$. The index of upper tail dependence is a concept relevant to the dependence in extreme values (which depends mainly on the tails) defined as follows.

Definition 2.2. Suppose a bivariate copula C is such that

$$\lambda_u = \lim_{v \to 1^-} \frac{1 - 2v + C(v, v)}{1 - v} \tag{2.8}$$

exists, then C has upper tail dependence if $\lambda_u \in (0, 1]$, and no upper tail dependence if $\lambda_u = 0$.

Remark 2.1. If $\{X_i, i \geq 1\}$ are identically distributed, Sklar's Theorem ensures us that $\lambda_u = 0$ is equivalent to the following equation.

$$\lim_{x \to \infty} \frac{\Pr(X_i > x, X_j > x)}{\Pr(X_i > x)} = 0, \quad \text{for } i \neq j, i, j \geq 1. \tag{2.9}$$

If the above condition (2.9) is satisfied, we say $\{X_i, i \geq 1\}$ is bivariate upper tail independent, or has no bivariate upper tail dependence, or has zero index of bivariate upper tail dependence.

Assumptions. For notational convenience, we state the following two assumptions regarding the sequences $\{X_i, i \geq 1\}$ and $\{\Theta_i, i \geq 1\}$.

H1. There exists a constant $\delta > 0$ such that $\mathsf{E}\Theta_i^{\beta+\delta} < \infty$ for all $i \geq 1$, and $\{X_i, i \geq 1\}$ are identically distributed as a generic c.d.f. $F \in ERV(-\alpha, -\beta)$, $0 < \alpha \leq \beta < \infty$, satisfying condition (2.9) and

$$\lim_{x \to \infty} \frac{\Pr(X < -x)}{\Pr(X > x)} = 0. \tag{2.10}$$

H2. $\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x) < \infty$ and either of the following two statements holds:

(1) for $0 < \beta < 1$, there exists $0 < \delta < \alpha$ such that $\beta + \delta < 1$ and

$$\sum_{i=1}^{\infty} \mathsf{E}\Theta_i^{\beta+\delta} < \infty, \qquad \sum_{i=1}^{\infty} \mathsf{E}\Theta_i^{\alpha-\delta} < \infty; \tag{2.11}$$

(2) for $1 \leq \beta < \infty$, there exists $0 < \delta < \alpha$ such that

$$\sum_{i=1}^{\infty} \left(\mathsf{E}\Theta_i^{\beta+\delta}\right)^{\frac{1}{\beta+\delta}} < \infty, \qquad \sum_{i=1}^{\infty} \left(\mathsf{E}\Theta_i^{\alpha-\delta}\right)^{\frac{1}{\beta+\delta}} < \infty. \tag{2.12}$$

In the proof of our main results (Theorem 3.1), we need the following series of lemmas. Besides their critical role in the proof of Theorem 3.1, some of these lemmas are themselves interesting. Among them, Lemma 2.1 is a direct consequence of Theorems 3.5(iii) and 3.5(v) in [8]. For the proof of the other lemmas, refer to Appendix A.2. It is worth noting that Davis and Resnick [10] have derived the same result as Lemma 2.2 for $\{X_i, i \geq 1\} \in \mathcal{R}_{-\alpha}$ with index $\alpha > 0$, and hence Lemma 2.2 is a generalized version of their result. The technique employed in the proof of Lemma 2.5 is a combination of that used in [23] and [13].

Lemma 2.1. *Suppose X is a nonnegative r.v. from $ERV(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, and Θ is a nonnegative r.v., for which there exists a constant $\delta > 0$ such that $\mathsf{E}\Theta^{\beta+\delta} < \infty$. Then $\Theta X \in ERV(-\alpha, -\beta)$.*

Lemma 2.2. *Suppose $\{X_i, i \geq 1\}$ is a sequence of nonnegative and identically distributed r.v.'s from $ERV(-\alpha, -\beta)$ such that (2.9) is satisfied. Then*

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \sim \sum_{i=1}^{n} \Pr(X_i > x) \tag{2.13}$$

holds for any fixed integer $n \geq 1$.

Lemma 2.3. *Suppose X_1 and X_2 are two identically distributed r.v.'s from $ERV(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, while Θ_1 and Θ_2 are another two nonnegative r.v.'s, independent of (X_1, X_2) and satisfying $\mathsf{E}\Theta_i^{\beta+\delta} < \infty$ for some $\delta > 0$, $i = 1, 2$. If X_1 and X_2 are bivariate upper tail independent, i.e. (2.9) holds with $i, j = 1, 2$, then*

$$\lim_{x\to\infty} \frac{\Pr(\Theta_1 X_1 > x, \Theta_2 X_2 > x)}{\Pr(\Theta_i X_i > x)} = 0, \quad \textit{for } i = 1, 2. \tag{2.14}$$

Lemma 2.4. *Suppose X is a r.v. from $ERV(-\alpha, -\beta)$ with $0 < \alpha \leq \beta < \infty$ such that (2.10) is satisfied, and Θ is another nonnegative r.v. satisfying $\mathsf{E}\Theta^{\beta+\delta} < \infty$ for some $\delta > 0$. Then*

$$\lim_{x\to\infty} \frac{\Pr(\Theta X < -xy)}{\Pr(\Theta X > x)} = 0 \tag{2.15}$$

holds for any $y > 0$.

Lemma 2.5. *Suppose $\{X_i, i \geq 1\}$ and $\{\Theta_i, i \geq 1\}$ are two sequences of r.v.'s satisfying assumptions* **H1** *and* **H2**, *then there exist a positive integer N_0 and a positive real number D such that, given any $\epsilon > 0$,*

$$\Pr\left(\sum_{i=n+1}^{\infty} \Theta_i X_i^+ > x\right) \leq (1+\epsilon) \sum_{i=n+1}^{\infty} \Pr(\Theta_i X_i > x) \tag{2.16}$$

holds for $n > N_0$ and $x > D$.

3. Main results and some remarks

The following theorem is our main result. We delay its proof to Appendix A.1.

Theorem 3.1. (a) *Suppose that assumption **H1** is satisfied for model (1.1), then we have for any fixed integer $n \geq 1$,*

$$\Pr\left(\max_{1\leq k\leq n} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) \sim \sum_{i=1}^{n} \Pr(\Theta_i X_i > x); \tag{3.1}$$

(b) *if additionally assumption **H2** holds, then we also have*

$$\Pr\left(\max_{1\leq k<\infty} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i^+ > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x). \tag{3.2}$$

Remark 3.1. Clearly, if $\{X_i, i \geq 1\}$ is a sequence of i.i.d. r.v.'s, condition (2.9) is satisfied automatically and hence can be dropped for the results in Theorem 3.1. Actually, if the sequence $\{X_i, i \geq 1\}$ are Negatively Quadrant Dependent (NQD) (see [17], p. 20 for the definition of NQD sequence), then condition (2.9) can also be dropped for Theorem 3.1, since in this case

$$0 \leq \lim_{x\to\infty} \frac{\Pr\{X_i > x, X_j > x\}}{\Pr(X_i > x)} \leq \lim_{x\to\infty} \frac{\Pr(X_i > x)\Pr(X_j > x)}{\Pr(X_i > x)} = 0, \quad i \neq j, i, j \geq 1. \tag{3.3}$$

It is worth mentioning that sequences of r.v.'s of those notions such as Negative Dependence (ND) (see [11], or [5]), and Negative Association (NA) (see [1], or [16]) all satisfy condition (2.9) by their relation with NQD.

Remark 3.2. If $F(x) \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then by the definition of $\mathcal{R}_{-\alpha}$, we see that (3.1) and (3.2) in Theorem 3.1 can be, respectively, rewritten as follows:

$$\Pr\left(\max_{1\leq k\leq n} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) \sim \overline{F}(x) \sum_{i=1}^{n} \mathsf{E}\Theta_i^{\alpha},$$

and

$$\Pr\left(\max_{1\leq k<\infty} \sum_{i=1}^{k} \Theta_i X_i > x\right) \sim \Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i^{+} > x\right) \sim \overline{F}(x) \sum_{i=1}^{\infty} \mathsf{E}\Theta_i^{\alpha}.$$

Moreover, when $\{X_i, i \geq 1\}$ are i.i.d r.v.'s from $\mathcal{R}_{-\alpha}$, condition (2.10) can be dropped for these asymptotic results to be valid. See Appendix A.3 for the proof. Thus, these results generalize the main results of [14] in the sense of allowing $\{X_i, i \geq 1\}$ to be generally dependent but with no bivariate upper tail dependence.

Remark 3.3. When $\{X_i, i \geq 1\}$ are identically distributed as a generic c.d.f. $F(x) \in \mathcal{R}_{-\alpha}$ for $\alpha > 0$, satisfying conditions (2.9) and (2.10), Weng et al. [30] established the following asymptotic results.

Let $\{r_i, i \geq 1\}$ be a sequence of constants such that $r_i \geq 0$, $i \geq 1$, and set $\theta_i = \prod_{j=1}^{i} (1 + r_j)^{-1}$, $i \geq 1$. Then

$$\Pr\left\{\max_{1\leq k\leq n} \sum_{i=1}^{k} \theta_i X_i > x\right\} \sim \Pr\left\{\sum_{i=1}^{n} \theta_i X_i > x\right\} \sim \overline{F}(x) \sum_{i=1}^{n} \theta_i^{\alpha}, \tag{3.4}$$

and if additionally $\sum_{i=1}^{\infty} \theta_i^{\alpha} < \infty$, then

$$\Pr\left\{\max_{1\leq k<\infty} \sum_{i=1}^{k} \theta_i X_i > x\right\} \sim \Pr\left\{\sum_{i=1}^{\infty} \theta_i X_i^{+} > x\right\} \sim \overline{F}(x) \sum_{i=1}^{\infty} \theta_i^{\alpha}. \tag{3.5}$$

Obviously, if taking $\Theta_i = \theta_i$ for each i in Theorem 3.1, we immediately recover (3.4) and (3.5) by the definition of $\mathcal{R}_{-\alpha}$. Weng et al. [30] also presented many comments on the conditions (2.9) and (2.10) in an actuarial perspective.

Remark 3.4. As pointed out in Remark 2.1, condition (2.9) holds if and only if the copula governing the bivariate dependence structure of the individual net losses has zero index of upper tail dependence. Consequently, to verify condition (2.9), one may turn to model the bivariate dependence structure of $\{X_i, i \geq 1\}$ via a bivariate copula, and to determine if the copula has zero index of upper tail dependence. For some families of copulas, the tail behavior is well known. For example, if the generator $\phi(t)$ for an Archimedean copula satisfies

$$\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t}\phi^{-1}(t) \neq -\infty, \tag{3.6}$$

where $\phi^{-1}(\cdot)$ is the inverse of $\phi(\cdot)$, then the copula has no upper tail dependence. See, for example, [22, Chapter 8] for detailed discussion on the Archimedean copulas with respect to the index of upper (lower) tail dependence. Other important families of copulas with zero index of upper tail dependence include Gaussian copulas, Farlie–Gumbel–Morgenstern copulas.

4. Applications of main results

In this section, we will present two examples to illustrate certain implications of our main results obtained in the previous section. In the first example, we will establish certain asymptotic bounds for the ruin probabilities in a discrete time risk model, while in the second one we will show some asymptotic properties with respect to the tail probability of the marginal of the stationary solution to a stochastic difference equation.

4.1. Ruin probabilities in a discrete time risk model

Let $Z_i \in (0, \infty)$ and $X_i \in (-\infty, \infty)$ be two r.v.'s, respectively, representing the discount factor and net loss for an insurance company, i.e. the total claim amount minus the total premium income, during the ith period, $i \geq 1$. The randomness of the discount factors may result from the stochastic interest rates or random return on investment in risky assets by the insurance company. Denote $\Theta_i = \prod_{j=1}^{i} Z_j$, $i \geq 1$. Then Θ_i stands for the discount factor from time i to time 0. Therefore, if we suppose that the net losses are calculated at the end of the year and that the insurance company starts with initial capital x, then her discounted surplus, denoted by U_n, accumulated till the end of year n is

$$U_0 = x, \qquad U_n = x - \sum_{i=1}^{n} X_i \Theta_i, \quad n \geq 1. \tag{4.1}$$

Set $S_n = \sum_{i=1}^{n} X_i \Theta_i$ for $n \geq 1$, then the finite time ruin probability is defined as

$$\psi(x; n) = \Pr\left(\max_{0\leq i\leq n} S_i > x | U_0 = x\right) \tag{4.2}$$

and the infinite time ruin probability as

$$\psi(x) = \lim_{n\to\infty} \psi(x; n) = \Pr\left(\max_{0\leq i<\infty} S_i > x | U_0 = x\right). \tag{4.3}$$

Note that $\psi(x; n)$ is the probability that the insurance company's surplus will become negative at certain time point during the period $[0, n]$, and $\psi(x)$ is the probability that the surplus will ever become negative on the infinite time horizon. Once the surplus becomes negative, we say that the insurance company gets ruined.

Risk models are basically of two classes: the continuous time models and the discrete time models. Discussing the ruin probabilities is one of the most important topics in risk theory. There are many publications tackling the ruin probabilities of model (4.1) and/or its variations. To our knowledge, however, all of them only address the case when $\{X_i, i \geq 1\}$ are independent just as introduced in the first section. Investigation on discrete time risk model (4.1) has many merits. The most important one is that the discrete time risk models themselves are interesting stochastic models both in theory and in application, and in many cases the associated ruin probabilities are crucial. Moreover, some continuous time risk models can be approximated by discrete time risk models, and ruin probabilities in many continuous time risk models can be reduced to those in embedded discrete time risk models. See, for example, [15,3,18], and references therein.

Now let us consider the ruin probabilities of model (4.1) by applying Theorem 3.1. For this purpose, we assume that the sequences $\{X_i, i \geq 1\}$ and $\{\Theta_i, i \geq 1\}$ satisfy corresponding conditions imposed in Theorem 3.1. Then, by the definitions of the above ruin probabilities, one immediately obtains the following asymptotics.

$$\psi(x, n) \sim \Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) \sim \sum_{i=1}^{n} \Pr(\Theta_i X_i > x), \tag{4.4}$$

and

$$\psi(x) \sim \Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i^+ > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x), \tag{4.5}$$

as the initial capital x tends to infinity.

With the moment conditions regarding Θ_i in Theorem 3.1, the assumptions in Theorem 3.3(iv) of [8] are satisfied. Consequently, by (4.4) and (4.5) we can further deduce the following asymptotic bounds for the ruin probabilities.

$$\overline{F}(x) \sum_{i=1}^{n} \mathsf{E} \min\left(\prod_{l=1}^{i} Z_l^{\alpha}, \prod_{l=1}^{i} Z_l^{\beta}\right) \lesssim \psi(x; n) \lesssim \overline{F}(x) \sum_{i=1}^{n} \mathsf{E} \max\left(\prod_{l=1}^{i} Z_l^{\alpha}, \prod_{l=1}^{i} Z_l^{\beta}\right), \tag{4.6}$$

$$\overline{F}(x) \sum_{i=1}^{\infty} \mathsf{E} \min\left(\prod_{l=1}^{i} Z_l^{\alpha}, \prod_{l=1}^{i} Z_l^{\beta}\right) \lesssim \psi(x) \lesssim \overline{F}(x) \sum_{i=1}^{\infty} \mathsf{E} \max\left(\prod_{l=1}^{i} Z_l^{\alpha}, \prod_{l=1}^{i} Z_l^{\beta}\right). \tag{4.7}$$

The above asymptotic results generalize the counterparts in Theorem 5.2 of [25] in the sense of allowing the net losses $\{X_i, i \geq 1\}$ to be r.v.'s that have no bivariate upper tail dependence while general dependence structure in other aspects.

If we assume that $\{Z_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative r.v.'s, then it is easy to check that the moment assumptions with respect to $\{\Theta_i, i \geq 1\}$ in Theorem 3.1 are equivalent to the following one:

H3. *There exists some* $\delta > 0$ *such that* $\mathsf{E}(Z_1^{\alpha+\delta}) < 1$.

Consequently, if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and assumption **H3** is satisfied, then applying Corollary 3.6 (iii) of [8] to the asymptotic formulas (4.4) and (4.5), we derive the following results:

$$\psi(x; n) \sim \overline{F}(x) \frac{EZ_1^{\alpha}(1 - (EZ_1^{\alpha})^n)}{1 - EZ_1^{\alpha}}, \tag{4.8}$$

$$\psi(x) \sim \overline{F}(x) \frac{EZ_1^{\alpha}}{1 - EZ_1^{\alpha}}. \tag{4.9}$$

It is worth mentioning that estimating ruin probabilities based on the risk models with dependent elements is one of the challenges in actuarial science. The results obtained so far are very limited, and only those risk modes with very special structures have been attempted; see, for example, [2,9].

4.2. Stochastic difference equations

Consider stochastic difference equations of the form

$$Y_n = Z_n Y_{n-1} + X_n, \quad n \geq 1, \tag{4.10}$$

with $\{X_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ being two sequences of r.v.'s, such that $\{Z_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$, $\{Z_n, n \geq 1\}$ are nonnegative i.i.d. r.v.'s, and $\{X_n, n \geq 1\}$ are identically distributed r.v.'s with no bivariate upper tail dependence.

Such equations have been widely studied in a variety of contexts under the assumption that $X_n, n \geq 1$ are identically and independently distributed; see, for example, [27,23] among many others. We are interested in the implications of Theorem 3.1 for such equations. If we further suppose that $\{X_n, n \geq 1\}$ are changeable, then the stationary solution $\{Y_n, n \geq 1\}$ of (4.10) exists with marginal distribution satisfying

$$Y_1 \stackrel{d}{=} \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} Z_i \right) X_n, \tag{4.11}$$

where an empty product is set equal to 1, and notation $\stackrel{d}{=}$ means equality in distribution. Now if we let $\Theta_n = \prod_{i=1}^{n-1} Z_i$ for $n \geq 1$, then the same reasoning as employed in the former subsection regarding ruin probabilities in model (4.1) will immediately enable us to obtain similar results for the tail probability $\Pr(Y_1 > x)$. Specifically, if $\{X_i, \Theta_i, i \geq 1\}$ satisfy the corresponding conditions in Theorem 3.1, then we have

$$\Pr(Y_1 > x) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x), \tag{4.12}$$

and

$$\overline{F}(x) \sum_{i=1}^{\infty} \mathsf{E} \min \left(\Theta_i^{\alpha}, \Theta_i^{\beta} \right) \lesssim \Pr(Y_1 > x) \lesssim \overline{F}(x) \sum_{i=1}^{\infty} \mathsf{E} \max \left(\Theta_i^{\alpha}, \Theta_i^{\beta} \right); \tag{4.13}$$

if $\{Z_n, n \geq 1\}$ are i.i.d. nonnegative r.v.'s, satisfying assumption **H3**, then

$$\Pr(Y_1 > x) \sim \frac{\overline{F}(x)}{1 - \mathsf{E} Z_1^{\alpha}}. \tag{4.14}$$

Note that (4.14) is the one-dimensional version of one result in [23], where $\{X_n, n \geq 1\}$ is assumed to be i.i.d. Thus the result we derived here is a partial generalization of theirs.

Acknowledgments

Weng thanks the fundings from the Institute for Quantitative Finance and Insurance, and the CAS/SOA Ph.D. grant. Zhang acknowledges the research support from the Institute for

Quantitative Finance and Insurance, and the National Natural Science Foundation of China (No. 10571159). All the authors are grateful to the two anonymous referees for their constructive comments and suggestions that make the paper significantly improved in both the content and presentation.

Appendix. Proof of main results and lemmas

A.1. Proof of main results

Proof of (a) of Theorem 3.1. Since $\{\Theta_i, i \geq 1\}$ are nonnegative, we have

$$\sum_{i=1}^{n} \Theta_i X_i \leq \max_{0 \leq k \leq n} \sum_{i=1}^{k} \Theta_i X_i \leq \sum_{i=1}^{n} \Theta_i X_i^+, \quad n \geq 1.$$

Thus, it suffices to show for $n \geq 1$,

$$\Pr\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x\right) \sim \sum_{i=1}^{n} \Pr(\Theta_i X_i > x), \tag{A.1}$$

and

$$\Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) \gtrsim \sum_{i=1}^{n} \Pr(\Theta_i X_i > x). \tag{A.2}$$

(A.1) follows immediately from the combination of Lemmas 2.1–2.3, and the fact that $\Pr\left(\Theta_i X_i^+ > x\right) = \Pr\left(\Theta_i X_i > x\right)$ for $x > 0$ and $i \geq 1$. Below we turn to prove (A.2).

Apparently, (A.2) holds for $n = 1$. Now we suppose $n \geq 2$. Let $v > 1$ be a constant, and set $y = (v-1)/(n-1)$. Then, clearly $y > 0$. For notation convenience, we further denote the events

$$A_i = (\Theta_i X_i \leq vx), \qquad A_i^- = (\Theta_i X_i \leq -yx), \quad i \geq 1.$$

In order to prove (A.2), we first analyze its left hand side.

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right) &\geq \Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x, \max_{1 \leq i \leq n} \Theta_i X_i > vx\right) \\
&\geq \sum_{i=1}^{n} \Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x, \Theta_i X_i > vx\right) \\
&\quad - \sum_{1 \leq k \neq l \leq n} \Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x, \Theta_k X_k > vx, \Theta_l X_l > vx\right) \\
&=: \Delta_1 - \Delta_2.
\end{aligned} \tag{A.3}$$

As for Δ_1 in (A.3), we see that

$$\begin{aligned}
&\Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x, \Theta_i X_i > vx\right) \\
&\quad \geq \Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x, \quad \Theta_i X_i > vx, \quad \Theta_j X_j > -yx \quad \text{for} \quad 1 \leq j \leq n, j \neq k\right)
\end{aligned}$$

$$= \Pr\left(\Theta_i X_i > vx,\, \Theta_j X_j > -yx,\, 1 \le j \le n,\, j \ne k\right)$$
$$\ge 1 - \left[\Pr(A_i) + \sum_{j=1, j\ne i}^{n} \Pr(A_j^-)\right], \quad 1 \le i \le n. \tag{A.4}$$

Moreover, it follows from Lemma 2.4 that

$$\lim_{x\to\infty} \frac{\Pr(A_j^-)}{\Pr(\Theta_j X_j > x)} = 0 \quad \text{for } 1 \le j \le n. \tag{A.5}$$

Hence, combining (A.4) and (A.5) yields

$$\liminf_{x\to\infty} \frac{\Delta_1}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)} \ge \liminf_{x\to\infty} \frac{\sum_{i=1}^{n}\left(\Pr(\Theta_i X_i > vx) - \sum_{j=1, j\ne i}^{n} \Pr(A_j^-)\right)}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)}$$
$$= \liminf_{x\to\infty} \frac{\sum_{i=1}^{n} \Pr\left(\Theta_i X_i > vx\right)}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)}. \tag{A.6}$$

As for Δ_2 in (A.3), by Lemma 2.3 we have

$$\lim_{x\to\infty} \frac{\Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x,\, \Theta_k X_k > vx,\, \Theta_l X_l > vx\right)}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)}$$
$$\le \lim_{x\to\infty} \frac{\Pr\left(\Theta_k X_k > vx,\, \Theta_l X_l > vx\right)}{\Pr(\Theta_l X_l > vx)} \limsup_{x\to\infty} \frac{\Pr\left(\Theta_l X_l > vx\right)}{\Pr(\Theta_l X_l > x)}$$
$$= 0, \quad \text{for } 1 \le k \ne l \le n,$$

which implies

$$\lim_{x\to\infty} \frac{\Delta_2}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)} = 0. \tag{A.7}$$

Finally, combining (A.3), (A.6) and (A.7), we obtain

$$\liminf_{x\to\infty} \frac{\Pr\left(\sum_{i=1}^{n} \Theta_i X_i > x\right)}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)} \ge \liminf_{x\to\infty} \frac{\sum_{i=1}^{n} \Pr\left(\Theta_i X_i > vx\right)}{\sum_{i=1}^{n} \Pr(\Theta_i X_i > x)}. \tag{A.8}$$

Letting $v \to 1$ in (A.8) leads to (A.2). $\square$

Proof of (b) of Theorem 3.1. It suffices to show

$$\Pr\left(\max_{1\le n<\infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x), \tag{A.9}$$

and

$$\Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i^+ > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x). \tag{A.10}$$

As the proof of (A.9) and (A.10) is quite similar, we shall just prove (A.9).

By (a) of Theorem 3.1, we have for any $m \geq 1$,

$$\Pr\left(\max_{1 \leq n < \infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \geq \Pr\left(\max_{1 \leq n \leq m} \sum_{i=1}^{n} \Theta_i X_i > x\right)$$
$$\sim \left(\sum_{i=1}^{\infty} - \sum_{i=m+1}^{\infty}\right) \Pr(\Theta_i X_i > x). \tag{A.11}$$

Since $\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x)$ is assumed to converge, letting $m \to \infty$ in (A.11) immediately yields

$$\Pr\left(\max_{1 \leq n < \infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \gtrsim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x).$$

Consequently, we will complete the proof if we can show

$$\Pr\left(\max_{1 \leq n < \infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \lesssim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x). \tag{A.12}$$

Note that

$$\max_{1 \leq n < \infty} \sum_{i=1}^{n} \Theta_i X_i \leq \max_{0 \leq n \leq m} \sum_{i=1}^{n} \Theta_i X_i + \sum_{i=m+1}^{\infty} \Theta_i X_i^+$$

holds for all $m \geq 1$. Hence, for any constants $0 < v < 1$, $m \geq 1$, and $x \geq 0$, we have

$$\Pr\left(\max_{1 \leq n < \infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \leq \Pr\left(\max_{0 \leq n \leq m} \sum_{i=1}^{n} \Theta_i X_i > (1-v)x\right)$$
$$+ \Pr\left(\sum_{i=m+1}^{\infty} \Theta_i X_i^+ > vx\right). \tag{A.13}$$

By (a) of Theorem 3.1, we have

$$\Pr\left(\max_{0 \leq n \leq m} \sum_{i=1}^{n} \Theta_i X_i > (1-v)x\right) \sim \sum_{i=1}^{m} \Pr\left(\Theta_i X_i > (1-v)x\right). \tag{A.14}$$

Moreover, given any $\epsilon > 0$, it follows from Lemma 2.5 that

$$\Pr\left(\sum_{i=m+1}^{\infty} \Theta_i X_i^+ > vx\right) \leq (1+\epsilon) \sum_{i=m+1}^{\infty} \Pr(\Theta_i X_i > vx) \tag{A.15}$$

holds for large enough integer m and real number x. Therefore, combining (A.13)–(A.15), we obtain

$$\Pr\left(\max_{1\le n<\infty}\sum_{i=1}^{n}\Theta_i X_i > x\right) \lesssim \sum_{i=1}^{m}\Pr(\Theta_i X_i > (1-v)x) + (1+\epsilon) \times \sum_{i=m+1}^{\infty}\Pr(\Theta_i X_i > vx). \tag{A.16}$$

Note that $\Theta_i X_i \in ERV(-\alpha, -\beta)$ for $i \ge 1$. Thus, if we first let $m \to \infty$ and then let $v \downarrow 0$ in the above equation, we immediately get (A.12), by which the proof is complete. □

A.2. *Proof of lemmas*

Proof of Lemma 2.2. Apparently, (2.13) holds for $n = 1$. Hence, we suppose $n \ge 2$.

On the one hand, we have

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \ge \Pr\left(\bigcup_{i=1}^{n}(X_i > x)\right) \ge \sum_{i=1}^{n}\Pr\left(X_i > x\right) - \sum_{1\le i\ne j\le n}\Pr\left(X_i > x, X_j > x\right),$$

and assumption (2.9) leads to

$$\frac{\sum_{1\le i\ne j\le n}\Pr\left(X_i > x, X_j > x\right)}{\sum_{i=1}^{n}\Pr(X_i > x)} \le \sum_{1\le i\ne j\le n}\frac{\Pr\left(X_i > x, X_j > x\right)}{\Pr(X_j > x)} \to 0.$$

Hence,

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \gtrsim \sum_{i=1}^{n}\Pr(X_i > x). \tag{A.17}$$

On the other hand, for any fixed real number v such that $1/2 < v < 1$,

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \le \Pr\left(\bigcup_{i=1}^{n}(X_i > vx)\right) + \Pr\left(\sum_{i=1}^{n} X_i > x, \bigcap_{j=1}^{n}(X_j \le vx)\right) =: I_1 + I_2. \tag{A.18}$$

As for I_1, we see by the arbitrariness of v on the interval $(1/2, 1)$ that

$$\lim_{v\nearrow 1}\limsup_{x\to\infty}\frac{I_1}{\sum_{i=1}^{n}\Pr(X_i > x)} \le \lim_{v\nearrow 1}\limsup_{x\to\infty}\frac{\sum_{i=1}^{n}\Pr\left(X_i > vx\right)}{\sum_{i=1}^{n}\Pr(X_i > x)} = 1. \tag{A.19}$$

Next we shall estimate I_2.

$$I_2 = \Pr\left(\sum_{i=1}^{n} X_i > x, \bigcap_{j=1}^{n}(X_j \le vx), \max_{1\le k\le n} X_k > \frac{x}{n}\right)$$

$$\leq \sum_{k=1}^{n} \Pr\left(\sum_{i=1}^{n} X_i > x, X_k \leq vx, X_k > \frac{x}{n}\right)$$

$$\leq \sum_{k=1}^{n} \Pr\left(\sum_{i=1, i\neq k}^{n} X_i > (1-v)x, X_k > \frac{x}{n}\right)$$

$$\leq \sum_{k=1}^{n} \sum_{i=1, i\neq k}^{n} \Pr\left(X_i > \frac{(1-v)x}{n-1}, X_k > \frac{x}{n}\right).$$

Since $\frac{1-v}{n-1} < 1$, we see that $\limsup_{x\to\infty} \frac{\Pr\left(X_k > \frac{(1-v)x}{n-1}\right)}{\Pr(X_k > x)} \leq \left(\frac{1-v}{n-1}\right)^{-\beta}$ for $1 \leq k \leq n$. Combining this with condition (2.9), we further have

$$\lim_{x\to\infty} \frac{I_2}{\sum_{i=1}^{n} \Pr(X_i > x)}$$

$$\leq \lim_{x\to\infty} \frac{\sum_{k=1}^{n} \sum_{i=1, i\neq k}^{n} \Pr\left(X_i > \frac{(1-v)x}{n-1}, X_k > \frac{x}{n}\right)}{\sum_{i=1}^{n} \Pr(X_i > x)}$$

$$\leq \lim_{x\to\infty} \sum_{k=1}^{n} \sum_{i=1, i\neq k}^{n} \frac{\Pr\left(X_i > \frac{(1-v)x}{n-1}, X_k > \frac{x}{n}\right)}{\Pr(X_k > x)}$$

$$\leq \lim_{x\to\infty} \sum_{k=1}^{n} \sum_{i=1, i\neq k}^{n} \frac{\Pr\left(X_i > \frac{(1-v)x}{n-1}, X_k > \frac{(1-v)x}{n-1}\right)}{\Pr\left(X_k > \frac{(1-v)x}{n-1}\right)} \frac{\Pr\left(X_k > \frac{(1-v)x}{n-1}\right)}{\Pr(X_k > x)}$$

$$= 0. \tag{A.20}$$

As a result, combining (A.18)–(A.20) yields

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \lesssim \sum_{i=1}^{n} \Pr(X_i > x). \tag{A.21}$$

Consequently, by (A.17) and (A.21), we obtain (2.13). □

Proof of Lemma 2.3. Due to the asymmetry between the two cases of $i = 1$ and $i = 2$, we will just prove (2.14) for $i = 2$ bellow.

Denote G_i to be the c.d.f. of Θ_i for $i = 1, 2$, and $G(x, y)$ to be the join distribution function of Θ_1 and Θ_2. Let us first analyze the numerator on the left hand side of (2.14).

$$\begin{aligned}
\Pr(\Theta_1 X_1 > x, \Theta_2 X_2 > x) &= \left(\int\int_{s\leq t} + \int\int_{s>t}\right) \Pr\left(X_1 > \frac{x}{s}, X_2 > \frac{x}{t}\right) \mathrm{d}G(s,t) \\
&\leq \int\int_{s\leq t} \Pr\left(X_1 > \frac{x}{t}, X_2 > \frac{x}{t}\right) \mathrm{d}G(s,t) \\
&\quad + \int\int_{s>t} \Pr\left(X_1 > \frac{x}{s}, X_2 > \frac{x}{s}\right) \mathrm{d}G(s,t) \\
&=: \Delta_1(x) + \Delta_2(x).
\end{aligned} \tag{A.22}$$

Since X_1 and X_2 belong to $ERV(-\alpha, -\beta)$,

$$\begin{aligned}\limsup_{x\to\infty}\frac{\Delta_1(x)}{\Pr(X_2>x)} &\le \limsup_{x\to\infty}\int_0^\infty \frac{\Pr\left(X_1>\frac{x}{t}, X_2>\frac{x}{t}\right)}{\Pr(X_2>x)}\mathrm{d}G_2(t)\\ &\le \int_0^\infty \limsup_{x\to\infty}\frac{\Pr\left(X_1>\frac{x}{t}, X_2>\frac{x}{t}\right)}{\Pr(X_2>x)}\mathrm{d}G_2(t)\\ &= 0. \end{aligned} \tag{A.23}$$

Moreover, due to the fact $\Pr(X_2\Theta_2 > x) \asymp \Pr(X_2 > x)$ (see Theorem 3.5(v) of [8]), (A.23) results in

$$\lim_{x\to\infty}\frac{\Delta_1(x)}{\Pr(X_2\Theta_2>x)} = 0. \tag{A.24}$$

Using the symmetry between $\Delta_1(x)$ and $\Delta_2(x)$, we also have

$$\lim_{x\to\infty}\frac{\Delta_2(x)}{\Pr(X_2\Theta_2>x)} = 0. \tag{A.25}$$

Consequently, combining (A.22), (A.24) and (A.25) results in (2.14). □

Proof of Lemma 2.4. The proof is quite similar to that of Lemma 2.3. Denote G to be the c.d.f of Θ, then for any $y > 0$,

$$\begin{aligned}\limsup_{x\to\infty}\frac{\Pr(\Theta X<-xy)}{\Pr(X>x)} &= \limsup_{x\to\infty}\int_0^\infty \frac{\Pr\left(X<-\frac{xy}{t}\right)}{\Pr(X>x)}\mathrm{d}G(t)\\ &\le \int_0^\infty \limsup_{x\to\infty}\frac{\Pr\left(X<-\frac{xy}{t}\right)}{\Pr(X>x)}\mathrm{d}G(t)\\ &= 0. \end{aligned} \tag{A.26}$$

Consequently, due to the fact that $\Pr(\Theta X > x) \asymp \Pr(X > x)$ we have

$$\limsup_{x\to\infty}\frac{\Pr(\Theta X<-xy)}{\Pr(\Theta X>x)} \le 0, \tag{A.27}$$

which immediately implies (2.15).

Proof of Lemma 2.5. Note that

$$\begin{aligned}&\Pr\left(\sum_{i=n+1}^\infty \Theta_i X_i^+ > x\right)\\ &\quad\le \Pr\left(\bigcup_{i=n+1}^\infty (\Theta_i X_i^+ > x)\right) + \Pr\left(\sum_{i=n+1}^\infty \Theta_i X_i^+ > x, \bigcap_{i=n+1}^\infty (\Theta_i X_i^+ \le x)\right)\\ &\quad\le \sum_{i=n+1}^\infty \Pr(\Theta_i X_i > x) + \Pr\left(\sum_{i=n+1}^\infty \Theta_i X_i^+ I_{(\Theta_i X_i^+\le x)} > x\right) \end{aligned} \tag{A.28}$$

holds for all $x \geq 0$ and any $n \geq 0$. Thus, it is sufficient for us to show

$$\lim_{n\to\infty} \frac{\Pr\left(\sum_{i=n+1}^{\infty} \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x)} > x\right)}{\sum_{i=n+1}^{\infty} \Pr\left(\Theta_i X_i^+ > x\right)} = 0 \tag{A.29}$$

holds for sufficiently large x. We shall consider the two cases in assumption **H2**.

Corresponding to the case (1) in assumption **H2**, we suppose temporarily that $0 < \beta < 1$, and that there exists $0 < \delta < \alpha$ satisfying $\beta + \delta < 1$, $\sum_{i=1}^{\infty} \mathsf{E}\,\Theta_i^{\beta+\delta} < \infty$, and $\sum_{i=1}^{\infty} \mathsf{E}\,\Theta_i^{\alpha-\delta} < \infty$. As for the generic r.v. X and the common c.d.f. $F(x)$ of $\{X_i, i \geq 1\}$, we see that $\Pr(X^+ \leq x) = 1 - \Pr(X^+ > x) = 1 - \Pr(X > x) = 1 - \overline{F}(x)$ and $x\overline{F}(x) \to \infty$ as $x \to \infty$. So, it follows from (2.6) that there exist positive constants C_2 and D_2 such that

$$\begin{aligned}
\frac{\mathsf{E}(X_i^+ I_{(X_i^+ \leq x)})}{x\overline{F}(x)} &= \frac{-x\overline{F}(x) + \int_0^x \overline{F}(s)\mathrm{d}s}{x\overline{F}(x)} \\
&= -1 + \left(\int_0^{D_2/x} + \int_{D_2/x}^1\right) \frac{\overline{F}(sx)}{\overline{F}(x)}\mathrm{d}s \\
&\leq -1 + o(1) + C_2 \int_0^1 s^{-(\beta+\delta)}\mathrm{d}s \\
&=: k_1 < \infty, \quad i \geq 1,
\end{aligned}$$

where $o(1)$ is in the sense that $x \to \infty$. Now we shall estimate the numerator on the left-hand side of (A.29). It follows from the Markov's inequality that

$$\begin{aligned}
\Pr\left(\sum_{i=n+1}^{\infty} \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x)} > x\right) &\leq \frac{1}{x}\left(\sum_{i=n+1}^{\infty} \mathsf{E}\left[\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\ 0 < \Theta_i \leq 1)}\right]\right) \\
&\quad + \frac{1}{x}\left(\sum_{i=n+1}^{\infty} \mathsf{E}\left[\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\ \Theta_i > 1)}\right]\right)
\end{aligned} \tag{A.30}$$

for $x \geq 0$ and $n \geq 0$. Let $G_k(t)$ be the c.d.f of Θ_k, $k \geq 1$. Then, using (2.5) we have

$$\begin{aligned}
\frac{\mathsf{E}\left[\Theta_i X_i^+ I_{(X_i^+ \Theta_i \leq x,\ 0 < \Theta_i \leq 1)}\right]}{x\overline{F}(x)} &= \int_0^1 \frac{\mathsf{E}[X_i^+ I_{(X_i^+ \leq x/t)}]}{x/t \cdot \overline{F}(x/t)} \cdot \frac{\overline{F}(x/t)}{\overline{F}(x)}\mathrm{d}G_i(t) \\
&\leq k_1 \int_0^1 \frac{1}{C_1} t^{\alpha-\delta}\mathrm{d}G_i(t) \\
&\leq \frac{k_1}{C_1}\mathsf{E}[\Theta_i^{\alpha-\delta}], \quad \text{for all } \delta > 0 \text{ and } i \geq 1.
\end{aligned} \tag{A.31}$$

Similarly, we can obtain for $i \geq 1$,

$$\begin{aligned}
\frac{\mathsf{E}\left(\Theta_i X_i^+ I_{(X_i^+ \Theta_i \leq x,\ \Theta_i > 1)}\right)}{x\overline{F}(x)} &= \int_1^{\infty} \frac{\mathsf{E}[X_i^+ I_{(X_i^+ \leq x/t)}]}{x/t \cdot \overline{F}(x/t)} \cdot \frac{\overline{F}(x/t)}{\overline{F}(x)}\mathrm{d}G_i(t) \\
&\leq C_2 k_1 \mathsf{E}\left(\Theta_i^{\beta+\delta}\right).
\end{aligned} \tag{A.32}$$

Moreover, we also have the fact as mentioned in the end of proof of Lemma 2.3 that $\Pr(\Theta_i X_i > x) \asymp \Pr(X_i > x)$, or equivalently,

$$\Pr(\Theta_i X_i^+ > x) \asymp \Pr(X_i > x), \quad i \geq 1. \tag{A.33}$$

Consequently, combining (A.30)–(A.33) and the assumptions $\sum_{i=1}^{\infty} \mathsf{E}(\Theta_i^{\beta+\delta}) < \infty$, $\sum_{i=1}^{\infty} \mathsf{E}(\Theta_i^{\alpha-\delta}) < \infty$, we have, for large enough x and any $n \geq 1$,

$$\begin{aligned}
&\frac{\Pr\left(\sum_{i=n+1}^{\infty} \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x)} > x\right)}{\sum_{i=n+1}^{\infty} \Pr\left(\Theta_i X_i^+ > x\right)} \\
&\quad \leq \left(\sum_{i=n+1}^{\infty} \frac{\mathsf{E}(\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\, \Theta_i \leq 1)})}{x \Pr\left(\Theta_i X_i^+ > x\right)}\right) + \left(\sum_{i=n+1}^{\infty} \frac{\mathsf{E}(\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\, \Theta_i > 1)})}{x \Pr\left(\Theta_i X_i^+ > x\right)}\right) \\
&\quad \leq E_1 \sum_{i=n+1}^{\infty} \mathsf{E}(\Theta_i^{\alpha-\delta}) + E_2 \sum_{i=n+1}^{\infty} \mathsf{E}(\Theta_i^{\beta+\delta}) \\
&\quad < \infty,
\end{aligned} \tag{A.34}$$

where E_1 and E_2 are some positive constants.

Now suppose, regarding (2) of assumption **H2**, that $1 \leq \beta < \infty$, and that there exists $0 < \delta < \alpha$ satisfying $\sum_{i=1}^{\infty} \left(\mathsf{E}\Theta_i^{\beta+\delta}\right)^{\frac{1}{\beta+\delta}} < \infty$ and $\sum_{i=1}^{\infty} \left(\mathsf{E}\Theta_i^{\alpha-\delta}\right)^{\frac{1}{\beta+\delta}} < \infty$. Then, by (2.6) there exists constants C_2 and D_2 such that

$$\begin{aligned}
\frac{\mathsf{E}\left[(X_i^+)^{\beta+\delta} I_{(X_i^+ \leq x)}\right]}{x^{\beta+\delta}\overline{F}(x)} &= \frac{-x^{\beta+\delta}\overline{F}(x) + \int_0^x (\beta+\delta) s^{\beta+\delta-1}\overline{F}(s)\mathrm{d}s}{x^{\beta+\delta}\overline{F}(x)} \\
&= -1 + \left(\mathrm{d}s \int_0^{D_2/x} + \mathrm{d}s \int_{D_2/x}^1\right) \frac{(\beta+\delta)s^{\beta+\delta-1}\overline{F}(sx)}{\overline{F}(x)}\mathrm{d}s \\
&\leq -1 + o(1) + C_2 \mathrm{d}s \int_0^1 (\beta+\delta) s^{\delta-\delta'-1}\mathrm{d}s \\
&=: k_2 < \infty
\end{aligned} \tag{A.35}$$

for any $0 < \delta' < \delta$ and $i \geq 1$, where $o(1)$ is in the sense that $x \to \infty$. Consequently, the same reasoning as in the case $0 \leq \beta < 1$ along with Minkovski inequality yields

$$\begin{aligned}
&\frac{\Pr\left(\sum_{i=n+1}^{\infty} \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x)} > x\right)}{\sum_{i=n+1}^{\infty} \Pr\left(\Theta_i X_i^+ > x\right)} \\
&\quad \leq \frac{\mathsf{E}\left[\sum_{i=n+1}^{\infty} \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x)}\right]^{\beta+\delta}}{x^{\beta+\delta} \sum_{i=n+1}^{\infty} \Pr\left(\Theta_i X_i^+ > x\right)}
\end{aligned}$$

$$
\leq \left[\sum_{i=n+1}^{\infty} \left(\frac{\mathsf{E}\left[\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\, \Theta_i \leq 1)}\right]^{\beta+\delta}}{x^{\beta+\delta} \Pr\left(\Theta_i X_i^+ > x\right)} \right)^{\frac{1}{\beta+\delta}} \right.
$$

$$
\left. + \sum_{i=n+1}^{\infty} \left(\frac{\mathsf{E}\left[\Theta_i X_i^+ I_{(\Theta_i X_i^+ \leq x,\, \Theta_i > 1)}\right]^{\beta+\delta}}{x^{\beta+\delta} \Pr\left(\Theta_i X_i^+ > x\right)} \right)^{\frac{1}{\beta+\delta}} \right]^{\beta+\delta}
$$

$$
\leq \left[E_1 \sum_{i=n+1}^{\infty} \left[\mathsf{E}\left(\Theta_i^{\alpha-\delta}\right)\right]^{\frac{1}{\beta+\delta}} + E_2 \sum_{i=n+1}^{\infty} \left[\mathsf{E}\left(\Theta_i^{\beta+\delta}\right)\right]^{\frac{1}{\beta+\delta}} \right]^{\beta+\delta}
$$

$$
< \infty \tag{A.36}
$$

for all $n \geq 1$, where E_1 and E_2 are some positive constants.

By (A.34) and (A.36), we see that (A.29) holds for both cases, and therefore the proof is complete.

A.3. Proof of Remark 3.2

Note that condition (2.10) is used in the proof of Theorem 3.1 only in derivation of (A.6). Suppose $\{X_i, i \geq 1\}$ are i.i.d. r.v.'s from $\mathcal{R}_{-\alpha}$. We shall show (A.6).

Based on (A.4), we see that for any $1 \leq i \leq n$,

$$
\begin{aligned}
&\Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x,\, \Theta_i X_i > vx\right) \\
&\quad \geq \Pr\left(\Theta_i X_i > vx,\, \Theta_s X_s > -yx,\, 1 \leq s \leq n,\, s \neq i\right) \\
&\quad = \mathsf{E}\left[\Pr\left(\Theta_i X_i > vx | \mathcal{F}_\Theta\right) \cdot \prod_{1 \leq s \leq n, s \neq i} \Pr\left(\Theta_s X_s > -yx | \mathcal{F}_\Theta\right)\right],
\end{aligned} \tag{A.37}
$$

where $\mathcal{F}_\Theta = \sigma\{\Theta_i, 1 \leq i \leq n\}$. As a result, applying Fatou Lemma and combining the fact that

$$
\lim_{x\to\infty} \Pr\{\Theta_s X_s > -yx | \mathcal{F}_\Theta\} = 1 \quad \text{for } 1 \leq s \leq n, \tag{A.38}
$$

we have

$$
\begin{aligned}
&\liminf_{x\to\infty} \frac{\Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x,\, \Theta_i X_i > vx\right)}{\Pr(\Theta_i X_i > x)} \\
&\quad \geq \mathsf{E}\left[\liminf_{x\to\infty} \frac{\Pr\left(\Theta_i X_i > vx | \mathcal{F}_\Theta\right)}{\Pr(\Theta_i X_i > x)} \cdot \prod_{1 \leq s \leq n, s \neq i} \lim_{x\to\infty} \Pr\left(\Theta_s X_s > -yx | \mathcal{F}_\Theta\right)\right] \\
&\quad = \mathsf{E}\left[\liminf_{x\to\infty} \frac{\Pr\left(\Theta_i X_i > vx | \mathcal{F}_\Theta\right)}{\Pr(\Theta_i X_i > x)}\right], \quad 1 \leq i \leq n.
\end{aligned} \tag{A.39}
$$

On the other hand, since $\{X_i, i \geq 1\}$ are from $\mathcal{R}_{-\alpha}$ and $\mathsf{E}\,\Theta_i^{\alpha+\delta} < \infty$, it follows from Lemma 2.1 that $\{\Theta_i X_i, i \geq 1\} \in \mathcal{R}_{-\alpha}$. Consequently, for any given $v > 1$,

$$\lim_{x\to\infty} \frac{\Pr(X_i > vx)}{\Pr(X_i > x)} = v^{-\alpha} < \infty, \quad 1 \leq i \leq n,$$

and

$$\lim_{x\to\infty} \frac{\Pr(\Theta_i X_i > x)}{\Pr(X_i > x)} = \mathsf{E}[\Theta_i^\alpha], \quad 1 \leq i \leq n.$$

Hence,

$$\begin{aligned} \mathsf{E}\left[\liminf_{x\to\infty} \frac{\Pr(\Theta_i X_i > vx|\mathcal{F}_\Theta)}{\Pr(\Theta_i X_i > x)}\right] &= \mathsf{E}\left[\liminf_{x\to\infty} \frac{\Pr(\Theta_i X_i > vx|\mathcal{F}_\Theta)}{\Pr(X_i > x)} \cdot \frac{\Pr(X_i > x)}{\Pr(\Theta_i X_i > x)}\right] \\ &= \mathsf{E}\left[\liminf_{x\to\infty} \frac{\Pr(\Theta_i X_i > vx|\mathcal{F}_\Theta)}{\Pr(X_i > x)}\right] \cdot \lim_{x\to\infty} \frac{\Pr(X_i > x)}{\Pr(\Theta_i X_i > x)} \\ &= \mathsf{E}[v^{-\alpha}\Theta_i^\alpha]/\mathsf{E}[\Theta_i^\alpha] \\ &= v^{-\alpha} < \infty, \quad 1 \leq i \leq n. \end{aligned} \tag{A.40}$$

Combining (A.39) and (A.40) leads to

$$\liminf_{x\to\infty} \frac{\Pr\left(\sum_{s=1}^{n} \Theta_s X_s > x, \Theta_i X_i > vx\right)}{\Pr(\Theta_i X_i > x)} \geq \lim_{x\to\infty} \frac{\Pr(\Theta_i X_i > vx)}{\Pr(\Theta_i X_i > x)}, \quad 1 \leq i \leq n, \tag{A.41}$$

by which we have (A.6) and the proof is complete. □

References

[1] K. Alam, K.M.L. Saxena, Positive dependence in multivariate distributions, Communications in Statistics — Theory and Methods 10 (1981) 1183–1196.

[2] H. Albrecher, Dependent risks and ruin probabilities in insurance, IIASA Interim Report, IR-98-072, 1998.

[3] S. Asmussen, Ruin Probabilities, World Scientific, Singapore, 2000.

[4] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.

[5] H.W. Block, T.H. Savits, M. Shaked, Some concepts of negative dependence, Annals of Probability 10 (1982) 765–772.

[6] Y. Chen, K.W. Ng, X. Xie, On the maximum of rnadomly weighted sums with regularly varying tails, Statistics and Probability Letters 76 (2006) 971–975.

[7] Y. Chen, C. Su, Finite time ruin probability with heavy-tailed insurance and financial risks, Statistics and Probability Letters 76 (2006) 1812–1820.

[8] D.B.H. Cline, G. Samorodnitsky, Subexponentiality of the product of independent random variables, Stochastic Processes and their Applications 49 (1994) 75–98.

[9] H. Cossette, É. Marceau, The discrete-time risk model with correlated classes of business, Insurance: Mathematics and Economics 26 (2000) 133–149.

[10] R.A. Davis, S.I. Resnick, Limit theory for bilinear processes with heavy- tailed noise, Annals of Applied Probability 6 (1996) 1191–1210.

[11] N. Ebrahimi, M. Ghosh, Multivariate negative dependence, Communications in Statistics — Theory and Methods 10 (1981) 307–337.

[12] P. Embrechts, C. Kluppelberg, T. Mikosch, Modelling Extremal Events for Insurance and Finance, Springer, Berlin, 1997.

[13] J.L. Geluk, C.G. De Vries, Weighted sums of subexponential random variables and asymptotic dependence between returns on reinsurance equities, Insurance: Mathematics and Economics 38 (2006) 39–56.

[14] M.J. Goovaerts, R. Kaas, R.J.A. Laeven, Q. Tang, R. Vernic, The tail probability of discounted sums of Pareto-like losses in insurance, Scandinavian Actuarial Journal (2005) 446–461.
[15] J. Grandell, Aspects of Risk Theory, Springer, New York, 1991.
[16] K. Joag-Dev, F. Proschan, Negative association of random variables, with applications, Annals of Statistics 11 (1983) 286–295.
[17] H. Joe, Multivariate Models and Dependence Concepts, Chapman & Hall, London, 1997.
[18] S. Klugman, H.H. Panjer, G.E. Willmot, Loss Models: From Data to Decisions, John Wiley & Sons, New York, NY, 2004.
[19] R.B. Nelsen, An Introduction to Copulas, 2nd edn, in: Springer Series in Statistics, Springer, New York, 2006.
[20] H. Nyrhinen, On the ruin probabilities in a general economic environment, Stochastic Processes and their Applications 83 (1999) 319–330.
[21] H. Nyrhinen, Finite and infinite time ruin probabilities in a stochastic economic environment, Stochastic Processes and their Applications 92 (2001) 265–285.
[22] H.H. Panjer, Operational Risk, John Wiley & Sons, New York, 2006.
[23] S.I. Resnick, E. Willehens, Moving averages with random coefficients and random coefficient autoregressive models, Communications in Statistics — Stochastic Models 7 (1991) 511–525.
[24] Q. Tang, Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation, Scandinivaian Actuarial Journal 1 (2005) 1–5.
[25] Q. Tang, G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks, Stochastic Processes and their Applications 108 (2003) 299–325.
[26] Q. Tang, G. Tsitsiashvili, Finite and infinite time ruin probabilities in the presence of stochastic return on investments, Advances in Applied Probability 36 (2004) 1278–1299.
[27] W. Vervaat, On a stochastic difference equation and a representation of non-negative infinitely divisible random variables, Advances in Applied Probability 11 (1979) 750–783.
[28] D. Wang, Q. Tang, Tail Probabilities of randomly weighted sums of random variables with variables with dominated variation, Stochastic Models 22 (2006) 253–272.
[29] D. Wang, C. Su, Y. Zeng, Uniform estimate for maximum of randomly weighted sums with applications to insurance risk theory, Science in China-Series A 48 (2005) 1379–1394.
[30] C. Weng, Y. Zhang, K.S. Tan, Ruin probability of a discrete time risk model under constant interest rate and dependent risks with heavy tails, manuscript, 2007.