Robustness in the Optimization of Risk Measures

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Agenda

1. Risk measures
2. Classic statistical robustness
3. Robustness in optimization
4. VaR and ES in representative optimization problems
5. Is distributionally robust optimization robust?
6. Conclusion

Based on joint work with Paul Embrechts (Zurich) and Alexander Schied (Waterloo)
Risk Measures

A risk measure $\rho : \mathcal{X} \to \mathbb{R} = (-\infty, \infty]$

- Risks are modelled by random losses in a specified period
  - e.g. 10d in Basel III & IV market risk
- $\mathcal{X}$ is a convex cone of rvs in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Roles of risk measures

- regulatory capital calculation ← our main interpretation
- management, optimization and decision making
- performance analysis and capital allocation
- risk pricing
General Question

Question

What is a “good” risk measure for regulatory capital calculation?

- Regulator’s and firm manager’s perspectives can be different or even conflicting
  - well-being of the society versus interest of the shareholders
  - systemic risk in an economy versus risk of a single firm
Value-at-Risk (VaR) at level $p \in (0, 1)$

$$\text{VaR}_p : L^0 \rightarrow \mathbb{R},$$

$$\text{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}.$$

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $p \in (0, 1)$

$$\text{ES}_p : L^0 \rightarrow \mathbb{R},$$

$$\text{ES}_p(X) = \frac{1}{1 - p} \int_p^1 \text{VaR}_q(X) dq = \mathbb{E}[X|X > \text{VaR}_p(X)].$$

$F_X$ above is the distribution function of $X$. 
Value-at-Risk and Expected Shortfall

density function or data histogram of $X$

$\text{VaR}_{0.99}(X)$

$\text{ES}_{0.99}(X)$
The ongoing **co-existence** of VaR and ES:

- Basel IV - **both**
- Solvency II - **VaR**
- Swiss Solvency Test - **ES**
ES is generally advocated by academia for desirable properties in the past two decades; in particular,

- subadditivity or coherence (Artzner-Delbaen-Eber-Heath’99)
- convex optimization properties (Rockafellar-Uryasev’00)

Some other examples of impact from academic research

- Gneiting’11: backtesting ES is unclear, whereas backtesting VaR is straightforward
- Cont-Deguest-Scandolo’10: ES is not robust, whereas VaR is
BCBS Consultative Document, May 2012, Page 41, Question 8:

“What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?”
### VaR versus ES

<table>
<thead>
<tr>
<th>Features/Risk measure</th>
<th>VaR</th>
<th>Tail-VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency captured?</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Severity captured?</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Sub-additive?</td>
<td>Not always</td>
<td>Always</td>
</tr>
<tr>
<td>Diversification captured?</td>
<td>Issues</td>
<td>Yes</td>
</tr>
<tr>
<td>Back-testing?</td>
<td>Straight-forward</td>
<td>Issues</td>
</tr>
<tr>
<td>Estimation?</td>
<td>Feasible</td>
<td>Issues with data limitation</td>
</tr>
<tr>
<td>Model uncertainty?</td>
<td>Sensitive to aggregation</td>
<td>Sensitive to tail modelling</td>
</tr>
<tr>
<td>Robustness I (with respect to “Lévy metric^33”)?</td>
<td>Almost, only minor issues</td>
<td>No</td>
</tr>
<tr>
<td>Robustness II (with respect to “Wasserstein metric^34“)?</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table copied from IAIS Consultation Document Dec 2014, page 42
Progress

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2. Classic statistical robustness

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4. VaR and ES in representative optimization problems

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VaR and ES are law-based (thus statistical risk functionals):
\[ \rho(X) = \rho(Y) \text{ if } X \overset{d}{=} Y \text{ (equal in distribution under } \mathbb{P}) \]

- The calculation requires knowledge of the distribution of a risk
- This may never be the exact case: model uncertainty
  - statistical error
  - computational error
  - modeling error
  - conceptual error

- Models are at most “approximately correct” \( \Rightarrow \) robustness!
Statistical robustness addresses the question of “what if the data is compromised with small error?”

- Originally **robustness** is defined on **estimators** (estimation procedures)
- Would the estimation be **ruined** if the underlying model is compromised?
  - e.g. an **outlier** is added to the sample
VaR and ES Robustness

density function or data histogram of $X$

$\text{VaR}_{0.99}(X)$

$\text{ES}_{0.99}(X)$

$\text{VaR}_{0.99}(X^*)$

$\text{ES}_{0.99}(X^*)$

single point huge value
VaR and ES Robustness

- Non-robustness of $\text{VaR}_p$ only happens if the quantile has a gap at $p$
- Is this situation relevant for risk management practice?
  - one must be very unlucky to hit precisely where it has a gap ...
Classic qualitative robustness:

- **Hampel’71**: the robustness of a consistent estimator of $T$ is equivalent to the continuity of $T$ with respect to underlying distributions (both with respect to the same metric).

- When we talk about the **robustness** of a statistical functional, (Huber-Hampel’s) robustness typically refers to continuity with respect to some metric.

- (Pseudo-)metrics: $\pi^q = L^q \ (q \geq 1)$, $\pi^\infty = L^\infty$, $\pi^W = \text{Lévy}, \ldots$

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General reference: Huber-Ronchetti’07

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Robustness in the optimization of risk measures
Consider the continuity of $\rho : X \to \mathbb{R}$.

- A strong sense of continuity is w.r.t. weak convergence.
  - $X_n \to X$ in distribution $\Rightarrow \rho(X_n) \to \rho(X)$.
- Quite restrictive
- Practitioners like weak convergence (e.g. estimation, simulation)
Robustness of Risk Measures

- With respect to weak convergence $p \in (0, 1)$:
  - $\text{VaR}_p$ is continuous at distributions whose quantile is continuous at $p$. $\text{VaR}_p$ is argued as being almost robust.
  - $\text{ES}_p$ is not continuous for any $\mathcal{X} \supset L^\infty$

- $\text{ES}_p$ is continuous w.r.t. some other (stronger) metric, e.g. $\pi^q$ (or the Wasserstein-$L^q$ metric)
Range-Value-at-Risk (RVaR)

A two-parameter family of risk measures, for $\alpha, \beta > 0$, $\alpha + \beta < 1$,

$$\text{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta} \int_{\alpha}^{\alpha+\beta} \text{VaR}_\gamma(X) d\gamma, \quad X \in \mathcal{X}.$$ 

- RVaR bridges the gap between VaR and ES (limiting cases).
- RVaR is continuous w.r.t. weak convergence
- RVaR is not convex or coherent
- Practically:

$$\text{RVaR}_{\alpha,\beta}(X) = \mathbb{E}[X | \text{VaR}_\alpha(X) < X \leq \text{VaR}_{\alpha+\beta}(X)]_{(F_X \text{ cont.})}.$$ 

First proposed by Cont-Deguest-Scandolo’10; name in W.-Bignozzi-Tsanakas’15
Range-Value-at-Risk (RVaR)

Distortion functions of $\text{VaR}_\alpha$ (red), $\text{ES}_\alpha$ (green) and $\text{RVaR}_{\alpha,\beta}$ (blue)

in the form of $\int_0^1 \text{VaR}_{\gamma}(X)dg(\gamma)$
Classic Robustness

The general perception of robustness, from worst to best:

\[ \text{ES} \prec \text{VaR} \prec \text{RVaR} \]

From weak to strong:

- Continuity w.r.t. \( \pi^\infty \): all monetary risk measures
- Continuity w.r.t. \( \pi^q \), \( q \geq 1 \): finite convex risk measures on \( L^q \), e.g. \( \text{ES}_p \)
- Continuity w.r.t. weak/a.s./\( P \) convergence: e.g. \( \text{RVaR}_{\alpha,\beta} \), \( \text{VaR}_p \) (almost); no convex risk measure satisfies this

Bäuerle-Müller’06, Cont-Deguest-Scandolo’10, Kou-Peng-Heyde’13
Is robustness w.r.t. weak convergence necessarily a good thing?

- **Toy example.**
  - Let \( X_n = n^2 \mathbf{1}_{\{U \leq 1/n\}} \) for some \( U[0,1] \) random variable \( U \) (e.g. a credit default risk). Clearly \( X_n \to 0 \) a.s. but \( X_n \) is getting more “dangerous” in many senses. If \( \rho \) preserves weak convergence, then

  \[
  \rho(X_n) \to \rho(0) \quad (=0 \text{ typically}).
  \]

  - \( \text{VaR}_{0.999}(X_{10000}) = 0 \)
  - \( \text{ES}_{0.999}(X_{10000}) = 10^7 \)

- May be reasonable for internal management; not so much for regulation.
One-in-ten-thousand Event

On the other hand,

- the 1/10,000-event-type risks are very difficult to capture statistically (accuracy is impossible)

UK House of Lords/House of Commons, June 12, 2013, Output of a “stress test” exercise, from HBOS:

“We actually got an external advisor [to assess how frequently a particular event might happen] and they came out with one in 100,000 years and we said “no”, and I think we submitted one in 10,000 years. But that was a year and a half before it happened. It doesn’t mean to say it was wrong: it was just unfortunate that the 10,000th year was so near.”
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Motivation

- So far, VaR and ES are applied to the same financial position.
- The regulatory choice of $\rho$ creates certain incentives, effective before $\rho$ is applied to assess risks.
- Once a specific $\rho$ has been chosen, portfolios will be optimized with respect to $\rho$ (at least to some extend).
- In reality, VaR and ES will not be applied to the same position.

One cannot decouple the technical properties of a risk measure from the incentives it creates.
The Optimization Problem

General setup

- $\mathcal{G}_n = \{\text{measurable functions from } \mathbb{R}^n \text{ to } \mathbb{R}\}$
- $X \in (L^0)^n$ is an economic vector, representing all random sources
- $\mathcal{G} \subset \mathcal{G}_n$ is a decision set
- $g(X)$ for $g \in \mathcal{G}$ represents a risky position of an investor
- $\rho$ is an objective functional mapping $\{g(X) : g \in \mathcal{G}\}$ to $\bar{\mathbb{R}}$

“The optimization problem”:

to minimize $\rho(g(X))$ over $g \in \mathcal{G}$
The Optimization Problem

Denote (possibly empty)

$$\mathcal{G}^*(X, \rho) = \left\{ g \in \mathcal{G} : \rho(g(X)) = \inf_{h \in \mathcal{G}} \rho(h(X)) \right\}.$$  

We call

- $g^* \in \mathcal{G}^*(X, \rho)$ an **optimizing function**

- $g^*(X)$ an **optimized position**
Uncertainty in Optimization

- The optimization problem is subject to model uncertainty
- Let $\mathcal{Z}$ be a set of possible economic vectors including $X$
  - $\mathcal{Z}$: the set of alternative models
  - e.g. a parametric family of models (parameter uncertainty)
- The true economic vector $Z \in \mathcal{Z}$ is likely different from the perceived economic vector $X$
  - $X$: best-of-knowledge model
  - $Z$: true model (unknowable)
- $g_X \in \mathcal{G}^*(X, \rho)$ is a best-of-knowledge decision
  - true position $g_X(Z)$
  - perceived position $g_X(X)$
Uncertainty in Optimization

We are interested in the **insolvency gap**

\[
\rho(g_X(Z)) - \rho(g_X(X))
\]

true risk - perceived risk

not the **optimality gap**

\[
\rho(g_Z(Z)) - \rho(g_X(Z))
\]

ture optimum - true risk

or the **difference between optima**

\[
\rho(g_Z(Z)) - \rho(g_X(X))
\]

ture optimum - perceived optimum
Uncertainty in Optimization

- If the modeling has good quality, $Z$ and $X$ are close to each other according to some metric $\pi$

- $\rho(g_X(Z))$ should be close to $\rho(g_X(X))$ to make sense of the optimizing function $g_X \Rightarrow$ some continuity of the mapping $Z \mapsto \rho(g_X(Z))$ at $Z = X$

- We call $(\mathcal{G}, \mathcal{Z}, \pi)$ an uncertainty triplet if $\mathcal{G} \subset \mathcal{G}_n$ and $(\mathcal{Z}, \pi)$ is a pseudo-metric space of $n$-random vectors.

- $\rho$ is compatible if its domain contains $\mathcal{G}(\mathcal{Z})$ and $\rho(g(Y)) = \rho(g(Z))$ for all $g \in \mathcal{G}$ and $Y, Z \in \mathcal{Z}$ with $\pi(Y, Z) = 0$. 
Robustness in Optimization

Definition 1

A compatible objective functional $\rho$ is robust at $X \in \mathcal{Z}$ relative to the uncertainty triplet $(\mathcal{G}, \mathcal{Z}, \pi)$ if there exists $g \in \mathcal{G}^*(X, \rho)$ such that the function $Z \mapsto \rho(g(Z))$ is $\pi$-continuous at $Z = X$.

- Robustness is a joint property of the tuple $(\rho, X, \mathcal{G}, \mathcal{Z}, \pi)$
- Only a $\pi$-neighbourhood of $X$ in $\mathcal{Z}$ matters
Remarks.

- If $\rho$ is robust at $X$ relative to $(\mathcal{G}, \mathcal{Z}, \pi)$, then it also holds
  - relative to $(\mathcal{G}, \mathcal{Z}', \pi)$ if $X \in \mathcal{Z}' \subset \mathcal{Z}$;
  - relative to $(\mathcal{G}, \mathcal{Z}, \pi')$ if $\pi'$ is stronger than $\pi$

- If $\mathcal{G}^*(X, \rho) = \emptyset$, then $\rho$ is not robust at $X$

- One can use topologies instead of metrics
- One can consider uncertainty on the set of probability measures instead of on the set of random vectors
- One can require the continuity for all $g \in \mathcal{G}^*(X, \rho)$ instead of that for some $g$. 
Progress

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4. **VaR and ES in representative optimization problems**
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Representative Optimization Problems

Representative optimization problems.

- \( n = 1 \) and \( X \geq 0 \) is a random loss
- The pricing density \( \gamma = \gamma(X) \) is a measurable function of \( X \)
  - \( \gamma > 0, \mathbb{E}[\gamma] = 1 \) and \( \mathbb{E}[\gamma X] < \infty \)
- The budget constraint is \( \mathbb{E}[\gamma g(X)] \geq x_0 \)
- Problems: to minimize \( \rho(g(X)) \) over \( g \in \mathcal{G} \) for some \( \mathcal{G} \subset \mathcal{G}_1 \)
  in three settings \( \mathcal{G} = \mathcal{G}_{cm}, \mathcal{G}_{ns}, \mathcal{G}_{bd} \)
Representative Optimization Problems

(a) Complete market:

\[ G_{cm} = \{ g \in G_1 : \mathbb{E}[\gamma g(X)] \geq x_0 \}. \]

(b) No short-selling or over-hedging constraint:

\[ G_{ns} = \{ g \in G_1 : \mathbb{E}[\gamma g(X)] \geq x_0, ~ 0 \leq g(X) \leq X \}. \]

Assume 0 \leq x_0 < \mathbb{E}[\gamma X] to avoid triviality.

(c) Bounded constraint: for some \( m > 0 \),

\[ G_{bd} = \{ g \in G_1 : \mathbb{E}[\gamma g(X)] \geq x_0, ~ 0 \leq g(X) \leq m \}. \]

Assume 0 \leq x_0 < m to avoid triviality.
Remark.

- Problem (c) is not a special case of Problem (b) as $X$ in (b) is both the constraint and the source of randomness.

For (a)-(c), assume

- The distribution function of $X$ is continuous and strictly increasing on $(\text{ess-inf} X, \text{ess-sup} X)$.

- $(\mathcal{Z}, \pi)$ is one of the classic choices ($L^q, \pi^q$) for $q \in [1, \infty]$ and ($L^0, \pi^W$), and $X \in \mathcal{Z}$.

We focus on $\text{VaR}_p$ and $\text{ES}_p$ for $p \in (0, 1)$.

Problem (c) for distortion risk measures: He-Zhou'11
Robustness in the Optimization of VaR

Let

\[ q = \inf \{ \text{VaR}_p(g(X)) : g \in \mathcal{G}_{ns} \}, \]

\[ q' = \inf \{ \text{VaR}_p(g(X)) : g \in \mathcal{G}_{bd} \}. \]

**Assumption 1**

\[ q > 0 \text{ and } \mathbb{P}((X - q)\gamma \leq \text{VaR}_p((X - q)\gamma)) = p. \]

**Assumption 2**

\[ q' > 0 \text{ and } \mathbb{P}(\gamma \leq \text{VaR}_p(\gamma)) = p. \]

- \( q, q' > 0 \) means the optimization does not result in zero risk
- Assumptions 1-2 are very weak
Solutions to the Representative Problems for $\text{VaR}_p$

Proposition 1 ($\text{VaR}_p$, Problem (c))

Let $U$ be a uniform transform of $\gamma$ on the probability space $(\Omega, \sigma(X), \mathbb{P})$.

(i) $q' = 0$ if and only if $\text{mES}_p(\gamma) \geq \frac{x_0}{1-p}$.

(ii) If $q' = 0$, a solution of Problem (c) is given by

$$g^*(X) = m 1_{\{U > p\}}.$$ 

(iii) If $q' > 0$, any solution to Problem (c) has the form

$$g^*(X) = m 1_{\{U > p\}} + q' 1_{\{U \leq p\}}, \text{ a.s.}$$
Robustness in the Optimization of VaR

Theorem 1

For $p \in (0, 1)$ and $X \in \mathcal{Z}$,

(i) $\text{VaR}_p$ is not robust relative to $(\mathcal{G}_{cm}, \mathcal{Z}, \pi)$;

(ii) under Assumption 1, $\text{VaR}_p$ is not robust at $X$ relative to $(\mathcal{G}_{ns}, \mathcal{Z}, \pi)$;

(iii) under Assumption 2, $\text{VaR}_p$ is not robust at $X$ relative to $(\mathcal{G}_{bd}, \mathcal{Z}, \pi)$.

Robustness of VaR in optimization is very bad.
Robustness in the Optimization of ES

Assumption 3

\[ \text{ess-sup} \gamma \leq \frac{1}{1-p}. \]

- Assumption 3 may be interpreted as a no-arbitrage condition for a market with ES participants.

Assumption 4

Either \( \gamma \) is a constant, or \( \gamma \) is a continuous function and \( \gamma(X) \) is continuously distributed.

- Assumption 4 is commonly satisfied.
Robustness in the Optimization of ES

Theorem 2

For $p \in (0, 1)$ and $X \in \mathcal{Z}$,

(i) under Assumption 3, $\text{ES}_p$ is robust at $X$ relative to $(\mathcal{G}_{cm}, \mathcal{Z}, \pi)$;

(ii) under Assumption 4, $\text{ES}_p$ is robust at $X$ relative to $(\mathcal{G}_{ns}, \mathcal{Z}, \pi)$ for $(\mathcal{Z}, \pi) = (L^q, \pi^q)$, $q \in [1, \infty]$;

(iii) under Assumption 4, $\text{ES}_p$ is robust at $X$ relative to $(\mathcal{G}_{bd}, \mathcal{Z}, \pi)$.

Robustness of ES in optimization is quite good
Robustness in Optimization for VaR and ES

On robustness in optimization:

\[ \text{VaR} \prec \prec \text{ES} \quad (\text{RVaR/ES not easy to compare}) \]

Observations.

- The discontinuity in \( Z \mapsto g^*(Z) \) comes from the fact that optimizing VaR is "too greedy": always ignores tail risk, and hoping the probability of the tail risk is correctly modelled.

- None of the two values

\[
\text{VaR}_p(g^*(X)) \quad \text{and} \quad \text{VaR}_p(g^*(Z))
\]

is a rational measure of the "optimized" risk.
Is risk positions of type $g^*$ realistic?

“Starting in 2006, the CDO group at UBS noticed that their risk-management systems treated AAA securities as essentially riskless even though they yielded a premium (the proverbial free lunch). So they decided to hold onto them rather than sell them.”

- From Feb 06 to Sep 07, UBS increased investment in AAA-rated CDOs by more than 10 times; many large banks did the same.
  - Take a risk of big loss with small probability
  - Treat it as free money - profit
  - Model uncertainty?

Quoted from Acharya-Cooley-Richardson-Walter’10
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Is Distributionally Robust Optimization Robust?

Distributionally robust optimization, for $\epsilon > 0$:

$$\text{to minimize: } \sup_{\pi(Y,X) \leq \epsilon} \rho(g(Y)) \quad \text{subject to } g \in \mathcal{G}.$$ 

- $\mathcal{G}^*(X, \rho, \epsilon)$: the set of functions $g \in \mathcal{G}$ solving this problem
- $\epsilon = 0$ leads to $\mathcal{G}^*(X, \rho, 0) = \mathcal{G}^*(X, \rho)$, the original setting
- $\rho$ is robust for the $\epsilon$-problem if there exists $g \in \mathcal{G}^*(X, \rho, \epsilon)$ such that $Z \mapsto \rho(g(Z))$ is $\pi$-continuous at $Z = X$
- This type of problems is hard to solve and we focus on VaR$_p$

for Problem (c): $(\mathcal{G}, Z, \pi) = (\mathcal{G}_{bd}, L^\infty, \pi^\infty)$.

- e.g. Natarajan-Pachamanova-Sim’08, Zhu-Fukushima’09, Ruodu Wang (wang@uwaterloo.ca)
The problem: to minimize

$$\sup_{\pi^\infty(Y,X) \leq \epsilon} \text{VaR}_p(g(Y))$$ subject to \(g \in G_{bd},\)

where \(G_{bd} = \{g \in G_1 : \mathbb{E}[\gamma g(X)] \geq x_0, \ 0 \leq g(X) \leq m\}.\) Let

$$q_\epsilon = \inf \left\{ \sup_{\pi^\infty(Y,X) \leq \epsilon} \text{VaR}_p(g(Y)) : g \in G_{bd} \right\}. $$

**Assumption 5**

\(q_\epsilon > 0, \ 1/2 \leq p < 1, \ X \ has \ a \ decreasing \ density \ on \)

\((\text{ess-inf} \ X, \text{ess-sup} \ X) \) \ and \ \(\gamma \ is \ an \ increasing \ function \ of \ X.\)
Proposition 2

Under Assumption 5, the above problem admits a solution of the form

\[ g^*(x) = m 1_{\{x>c+\epsilon\}} + q_\epsilon 1_{\{x\leq c+\epsilon\}}, \ x \in \mathbb{R}, \text{ where } c = \text{VaR}_p(X). \]

- \( Z \mapsto \text{VaR}_p(g^*(Z)) \) is \( \pi^\infty \)-continuous at \( Z = X \)
- \( \text{VaR}_p \) is robust for the \( \epsilon \)-problem
- The \( \epsilon \)-modification improves the robustness of VaR
- We still get the big-loss-small-probability type of optimizer
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Some conclusions on robustness

- **Classic notion**
  - \( \text{ES} \preceq \text{VaR} \preceq \text{RVaR} \)
  - However this robustness may not be desirable

- **If we take optimization into account**
  - \( \text{VaR} \not\preceq \text{ES} \) in optimization
  - The rationality of optimizing VaR under model uncertainty is questionable

- **Some other perspectives**
  - \( \text{VaR} \preceq \text{ES} \preceq \text{RVaR} \) in risk aggregation
  - \( \text{VaR} \not\preceq \text{ES} \preceq \text{RVaR} \) in risk sharing

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Embrechts-Wang-W.'15, Krätschmer-Schied-Zähle'17, Embrechts-Liu-W.'18
Other Questions

Many other questions ...

- other risk measures
- other optimization problems
- utility maximization problems
- risk measures as constraints instead of objectives
CEOs of AIG Financial Products, August 2007:

“It is hard for us, without being flippant, to even see a scenario within any kind of realm of reason that would see us losing one dollar in any of those transactions.”

▶ AIGFP sold protection on super-senior tranches of CDOs
▶ $180 billion bailout from the federal government in September 2008
Thank You

This paper is available on SSRN (3254587) and arXiv (1809.09268)
Proposition 3 \((\text{VaR}_p, \text{Problem (a)})\)

\[
\inf\{\text{VaR}_p(g(X)) : g \in \mathcal{G}_{\text{cm}}\} = -\infty. \quad \text{Hence, Problem (a) admits no solution.}
\]
Solutions to the Representative Problems for $\text{VaR}_p$

**Proposition 4 ($\text{VaR}_p$, Problem (b))**

Let $U$ be a uniform transform of $(X - q)\gamma$ on the probability space $(\Omega, \sigma(X), \mathbb{P})$.

(i) $q = 0$ if and only if $\text{ES}_p(\gamma X) \geq \frac{x_0}{1-p}$.

(ii) If $q = 0$, a solution of Problem (b) is given by

$$g^*(X) = X1_{\{U > p\}}.$$

(iii) If $q > 0$, any solution to Problem (b) has the form

$$g^*(X) = X1_{\{U > p\}} + (X \wedge q)1_{\{U \leq p\}}, \text{ a.s.}$$
Proposition 5 \((\text{VaR}_p, \text{Problem (c)})\)

Let \(U\) be a uniform transform of \(\gamma\) on the probability space \((\Omega, \sigma(X), \mathbb{P})\).

(i) \(q' = 0\) if and only if \(m \mathbb{E} S_p(\gamma) \geq \frac{x_0}{1-p}\).

(ii) If \(q' = 0\), a solution of Problem (c) is given by

\[ g^*(X) = m \mathbb{1}_{U > p}. \]

(iii) If \(q' > 0\), any solution to Problem (c) has the form

\[ g^*(X) = m \mathbb{1}_{U > p} + q' \mathbb{1}_{U \leq p}, \text{ a.s.} \]
Proposition 6 ($\text{ES}_p$, Problem (a))

Problem (a) admits a solution if and only if Assumption 3 holds, and if Assumption 3 holds, a solution is given by $g^*(\cdot) = x_0$. 
Proposition 7 (\( \text{ES}_p \), Problem (b))

There exist constants \( c > 0, r \geq 0 \), and \( \lambda \in [0, 1] \) such that the function \( g^* \), for \( x \in \mathbb{R} \),

\[
g^*(x) = x 1_{\{\gamma(x) > c\}} + (x \wedge r) 1_{\{\gamma(x) < c\}} + ((1-\lambda)x + \lambda (x \wedge r)) 1_{\{\gamma(x) = c\}},
\]

solves Problem (b). Moreover, \( r \) is a \( p \)-quantile of \( g^*(X) \).
Proposition 8 ($\text{ES}_p$, Problem (c))

There exist constants $c > 0$, $r \in [0, m]$, and $\lambda \in [r, m]$ such that the function $g^*$, for $x \in \mathbb{R}$,

$$g^*(x) = m \mathbb{1}_{\{\gamma(x) > c\}} + r \mathbb{1}_{\{\gamma(x) < c\}} + \lambda \mathbb{1}_{\{\gamma(x) = c\}},$$

solves Problem (c). Moreover, $r$ is a $p$-quantile of $g^*(X)$. 
From the **International Association of Insurance Supervisors:**

- Document (version June 2015)
  Compiled Responses to ICS Consultation 17 Dec 2014 - 16 Feb 2015

**In summary**

- Responses from insurance organizations and companies in the world.
- 49 responses are public
- 34 commented on Q42: VaR versus ES (TVaR)
Industry Perspectives

- 5 responses are supportive about ES:
  - Canadian Institute of Actuaries, CA
  - Liberty Mutual Insurance Group, US
  - National Association of Insurance Commissioners, US
  - Nematrian Limited, UK
  - Swiss Reinsurance Company, CH

- Some are indecisive; most favour VaR.

Regulator and firms may have different views
Major reasons to favour VaR from the insurance industry (IAIS report June 2015)

- Implementation of ES is expensive (staff, software, capital)
- ES does not exist for certain heavy-tailed risks
- ES is more costly on distributional information, data and simulation
- ES has trouble with a change of currency