A Theory for Measures of Tail Risk

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Based on joint work with Fangda Liu (CUFE, Beijing) and joint work with Edward Furman (York, Toronto) and Ričardas Zitikis (Western, London Ontario)
A risk measure ...

- quantifies the “riskiness” of a random loss in a fixed period of time (e.g. 1 year, 10 days, 1 day)
- primary examples in practice: the Value-at-Risk and the Expected Shortfall
- main interpretation: the amount of regulatory capital of a financial institution taking a risk (random loss) in a fixed period
Value-at-Risk and Expected Shortfall

For $X \in L^0$, the Value-at-Risk (VaR) at confidence level $p \in (0, 1)$ has two versions:

$$\text{VaR}_L^p(X) = \inf \{ x \in \mathbb{R} : F_X(x) \geq p \} = F_X^{-1}(p),$$

and

$$\text{VaR}_R^p(X) = \inf \{ x \in \mathbb{R} : F_X(x) > p \} = F_X^{-1}(p^+).$$

The Expected Shortfall (ES) at confidence level $p \in (0, 1)$:

$$\text{ES}_p(X) = \frac{1}{1 - p} \int_p^1 \text{VaR}_q^L(X) dq = \frac{1}{1 - p} \int_p^1 \text{VaR}_q^R(X) dq$$

▷ Typical choices of $p$: 0.975, 0.99, 0.995, 0.999 ...
Value-at-Risk and Expected Shortfall

\[
\begin{align*}
\text{VaR}_p^R(X) & \quad \text{VaR}_p^L(X) \\
\text{ES}_p(X) & \quad F_X^{-1}(t)
\end{align*}
\]
The ongoing debate on “VaR versus ES”:

- Basel III (mixed; in transition from VaR to ES as standard metric for market risk)
- Solvency II (VaR based)
- Swiss Solvency Test (ES based)

Some academic references

- Embrechts-Puccetti-Rüschendorf-W.-Beleraj 2014
- Emmer-Kratz-Tasche 2015

“... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of “tail risk” and capital adequacy during periods of significant financial market stress.”

Some interpretation:

- “tail risk” is a crucial concern for prudent risk management
- “tail risk” is associated with financial market stress
So ... what is “tail risk”? 

**Tail risk** is a form of portfolio risk that arises when the possibility that an investment will move more than three standard deviations from the mean is greater than what is shown by a normal distribution.

*Tail Risk - Investopedia*
[www.investopedia.com/terms/t/tailrisk.asp](http://www.investopedia.com/terms/t/tailrisk.asp)

**Tail risk** is the additional risk of an asset or portfolio of assets moving more than 3 standard deviations from its current price, above the risk of a normal distribution.

*Tail risk - Wikipedia*
Tail risk

That is wrong on so many levels.

– Apple’s Siri, as quoted by Sidney Resnick, April 2017, Zurich

- Probability of moving downside more than 3 standard deviations:
  - normal risk: 0.135%
  - Pareto(5) risk: 1.86%
  - Pareto(3) risk: 1.45%
  - Pareto(2.01) risk: 0.05%
  - Cantalli’s inequality: $\leq 10\%$ ($10\% \Rightarrow$ Bernoulli)

- No tail risk for very heavy-tailed distributions?

- A Bernoulli distribution has the most severe tail risk?
Tail risk

Our motivation

- establish a framework for regulatory concerns of tail risks
- complementary risk metrics to VaR and ES
  - VaR and ES are similar to “median” and ”mean”
- alternative risk measures (internal risk management)
- better understand the roles of VaR and ES
Progress of the talk

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Notation

Some notation.

- \((\Omega, \mathcal{F}, \mathbb{P})\) is an atomless probability space
- \(\mathcal{X}\) is a convex cone of random variables containing \(L^\infty\)
  - e.g. \(\mathcal{X} = L^\infty\)
- A risk measure is a functional \(\rho : \mathcal{X} \to (-\infty, \infty]\) such that \(\rho(X) \in \mathbb{R}\) for \(X \in L^\infty\).
- For \(X \in \mathcal{X}\),
  - \(F_X\): cdf of \(X\)
  - \(F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}, p \in (0, 1]\)
  - \(U_X\): a uniform random variable satisfying \(F_X^{-1}(U_X) = X\) a.s.\(^1\)

\(^1\)such \(U_X\) always exists; see Lemma A.32 of Föllmer-Schied 2016
Tail risk

For any random variable $X \in \mathcal{X}$ and $p \in (0, 1)$, let $X_p$ be the tail risk of $X$ beyond its $p$-quantile

$$X_p = F_X^{-1}(p + (1 - p)U_X).$$

- $p + (1 - p)U_X$ is uniform on $[p, 1]$. 
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Tail risk

\[ \text{VaR}_p^R(X) \]

\[ F_X^{-1}(t) \]

\[ F_{X_p}^{-1}(t) \]
Tail risk measures

Definition 1

For \( p \in (0, 1) \), a risk measure \( \rho \) is a \( p \)-tail risk measure if \( \rho(X) = \rho(Y) \) for all \( X, Y \in \mathcal{X} \) satisfying \( X_p \overset{d}{=} Y_p \).

For \( p \in (0, 1) \),

- \( \text{ES}_p \) is a \( p \)-tail risk measure
- \( \text{VaR}^R_p \) is a \( p \)-tail risk measure
- \( \text{VaR}^L_p \) is not a \( p \)-tail risk measure, but a \((p - \epsilon)\)-tail risk measure for all \( \epsilon \in (0, p) \)
- a \( p \)-tail risk measure is law-invariant

(A0) Law-invariance: \( \rho(X) = \rho(Y) \), if \( X \overset{d}{=} Y, X, Y \in \mathcal{X} \).
Tail risk

\[ F_X^{-1}(t) \]

\[ \text{ES}_p(X) \]

\[ \text{VaR}_p^R(X) \]

\[ \text{VaR}_p^L(X) \]

\[ \mathbb{E}[X_p] \]

\[ \text{ess-inf}(X_p) \]
Generators of tail risk measures

Observe simple relations

\[ \text{VaR}_p^R(X) = \text{ess-inf}(X_p) \quad \text{and} \quad \text{ES}_p(X) = \mathbb{E}[X_p], \quad X \in \mathcal{X}. \]

Generally, for any law-invariant risk measure \( \rho^* \) on \( \mathcal{X} \), define

\[ \rho(X) = \rho^*(X_p), \quad X \in \mathcal{X}. \]

then \( \rho \) is a \( p \)-tail risk measure.

- \( \rho \) is \text{generated by} \( \rho^* \) and \( \rho^* \) is a \( p \)-\text{generator} of \( \rho \).
- There is a \text{one-to-one} relationship between \( \rho \) and \( \rho^* \).
Tail pair of risk measures

A pair of risk measures \((\rho, \rho^*)\) is called a \(p\)-tail pair if \(\rho^*\) is law-invariant and is a \(p\)-generator of \(\rho\).

Examples.

- \((\text{VaR}_p^R, \text{ess-inf})\)
- \((\text{VaR}^R_{(p+1)/2}, \text{right-median})\)
- \((\text{ES}_p, \mathbb{E})\)
- \((\text{VaR}^L_q, \text{VaR}^L_q/(1-p)), q > p\)
Properties of tail risk measures

Estimation.

To estimate a tail risk measure
  ▶ estimate the tail distribution (distribution of $X_p$; maybe through Extreme Value Theory if $p$ is close to 1)
  ▶ apply a standard procedure to estimate the generator $\rho^*(X_p)$

Question.

How do we generate a tail risk measure with desirable properties?
  ▶ what properties may be passed from $\rho^*$ to $\rho$?

For instance, if $\rho^*$ is convex, is $\rho$ also convex?
Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Monetary risk measures

(A1) **Monotonicity**: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

(A2) **Translation invariance**: $\rho(X + m) = \rho(X) + m$ if $m \in \mathbb{R}$.

Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Convex risk measures

(A1) **Monotonicity**: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

(A2) **Translation invariance**: $\rho(X + m) = \rho(X) + m$ if $m \in \mathbb{R}$.

(A3) **Convexity**: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 + \lambda)\rho(Y)$ for $\lambda \in [0, 1]$.

Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Coherent risk measures

(A1) Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

(A2) Translation invariance: $\rho(X + m) = \rho(X) + m$ if $m \in \mathbb{R}$.

(A3) Convexity: $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 + \lambda) \rho(Y)$ for $\lambda \in [0, 1]$.

(A4) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda > 0$.

(A5) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

More properties

(A6) **Comonotonic additivity:** \( \rho(X + Y) = \rho(X) + \rho(Y) \) if \( X, Y \in \mathcal{X} \) are comonotonic.

(A7) **\( \prec_{cx} \)-monotonicity:** \( \rho(X) \leq \rho(Y) \) if \( X \prec_{cx} Y \).

**Notes.** For \( X, Y \in \mathcal{X} \),

- \( X, Y \) are comonotonic if \( (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \) for all \( (\omega, \omega') \in \Omega \times \Omega \), \( \mathbb{P} \times \mathbb{P} \)-almost surely.
- \( Y \) is smaller than \( X \) in **convex order**, denoted as \( X \prec_{cx} Y \), if \( X \overset{d}{=} \mathbb{E}[Y|G] \) for some \( \sigma \)-algebra \( G \subset \mathcal{F} \); equivalently, \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \) (if both exist) for all convex \( f \).
VaR and ES

Take $p \in (0, 1)$.

- For $(\rho, \rho^*) = (\text{VaR}_p^R, \text{ess-inf})$, both $\rho$ and $\rho^*$ are monotone, translation-invariant, positively homogeneous, and comonotonically additive.

- For $(\rho, \rho^*) = (\text{ES}_p, \mathbb{E})$, both $\rho$ and $\rho^*$ are, in addition to the above, subadditive, convex, and $\prec_{\text{cx}}$-monotone.
Tail standard deviation risk measure

Take \( p \in (0, 1) \), \( X = L^2 \) and let \( \rho^* \) be the standard deviation risk measure for some \( \beta > 0 \)

\[
\rho^*(X) = \mathbb{E}[X] + \beta \sqrt{\text{var}(X)}, \quad X \in L^2.
\] (1)

Let

\[
\rho(X) = \rho^*(X^p) = \mathbb{E}[X^p] + \beta \sqrt{\text{var}(X^p)}, \quad X \in L^2.
\]

- \( \rho^* \) is convex, subadditive, and \( \prec_{\text{cx}} \)-monotone
- but \( \rho \) is NOT convex, subadditive, or \( \prec_{\text{cx}} \)-monotone!
Properties of tail risk measures

Theorem 2
Suppose that $p \in (0, 1)$ and $(\rho, \rho^*)$ is a $p$-tail pair of risk measures on $\mathcal{X}$ and $\mathcal{X}^*$, respectively. The following statements hold.

(i) $\rho$ is monotone (translation-invariant, positively homogeneous, comonotonically additive) if and only if so is $\rho^*$.

(ii) If $\rho$ is subadditive (convex, $\prec_{cX}$-monotone) then so is $\rho^*$.

Remark.

- The converse of (ii) is not true because of the previous example.
Properties of tail risk measures

**Theorem 3**

*Suppose that $p \in (0, 1)$ and $(\rho, \rho^*)$ is a $p$-tail pair of risk measures on $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Then $\rho$ is a coherent (convex, monetary) risk measure if and only if so is $\rho^*$.***

**Remark.**

- Monotonicity is essential for the other properties to pass through.
- To construct coherent tail risk measures: apply an existing coherent risk measure to the tail risk $X_p$. 
Smallest tail risk measures

Theorem 4

For \( p \in (0, 1) \), if \( \rho \) is a monetary \( p \)-tail risk measure with \( \rho(0) = 0 \), then \( \rho \geq \text{VaR}_p^R \) on \( \mathcal{X} \), and if \( \rho \) is a coherent \( p \)-tail risk measure, then \( \rho \geq \text{ES}_p \) on \( \mathcal{X} \).

Remark.

- VaRs and ES serve as benchmarks for tail risk measures.
- The converse statements are not true in general. For instance, take \( \rho(X) = \max \{ \mathbb{E}[X], \text{VaR}_p^R(X) \} \).
- For distortion risk measures, the converse statements are true.

Notes.

- A distortion risk measure is a law-invariant, comonotonic-additive and monetary risk measure.
Other properties

Other mathematical or statistical properties:

- **Superadditivity and concavity.** Counter-example: \((\text{VaR}^R_p, \text{ess-inf})\)
- **Linearity.** Counter-example: \((\text{ES}_p, \mathbb{E})\)
- **Common robustness (continuity) properties** such as continuity with respect to Wasserstein \(L^q\)-norm \((q \geq 1)\) or convergence in distribution can be naturally passed on from \(\rho^*\) to \(\rho\).
- **Elicitability** cannot be passed from \(\rho^*\) to \(\rho\) (will be discussed later)
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C. Gini noticed the center-free version of the variance

\[ \text{Var}(X) = \frac{1}{2} \mathbb{E}[(X^* - X^{**})^2], \quad X \in L^2, \]

where \( X^* \) and \( X^{**} \) are two independent copies of \( X \).

Consequently, he introduced

\[ \text{Gini}(X) = \mathbb{E}[|X^* - X^{**}|], \quad X \in L^1, \]

which is nowadays known as the Gini mean difference.

The Gini functional is comonotonic-addtive and satisfies

\[ \text{Gini}(X) = 2 \int_0^1 F_X^{-1}(u)(2u - 1)du = 4 \text{Cov}(X, U_X). \]
D. Denneberg, insisting comonotonic-additivity, introduced the **Gini principle** to replace the standard deviation risk measure, for \( \lambda > 0 \).

\[
\text{GP}^\lambda(X) = \mathbb{E}[X] + \lambda \text{Gini}(X), \quad X \in L^1.
\]

Using the language of risk measures, the Gini principle is convex, subadditive, \( \prec_{cx} \)-monotone, positively homogeneous, comonotonic-additive and translation invariant, but not necessarily monotone.
By applying the Gini principle to the tail risk, define the Gini Shortfall for \( p \in [0, 1) \) and \( \lambda > 0 \),

\[
\text{GS}_p^\lambda(X) = \mathbb{E}[X_p] + \lambda \text{Gini}(X_p), \quad X \in L^1,
\]

and equivalently,

\[
\text{GS}_p^\lambda(X) = \text{ES}_p(X) + \lambda \mathbb{E}[|X_p^* - X_p^{**}|], \quad X \in L^1,
\]

where \( X_p^* \) and \( X_p^{**} \) are two independent copies of \( X_p \).

- A Gini Shortfall combines magnitude (captured by the ES part) and variability (captured by the Gini part) of tail risk.
Gini Shortfall

Theorem 5

Let \( p \in (0, 1) \) and \( \lambda \in [0, \infty) \).

1. The functional \( GS^\lambda_p \) is translation invariant, positively homogeneous, and comonotonic-additive.

2. The following statements are equivalent:
   
   (i) \( GS^\lambda_p \) is monotone;
   
   (ii) \( GS^\lambda_p \) is convex;
   
   (iii) \( GS^\lambda_p \) is subadditive;
   
   (iv) \( GS^\lambda_p \) is \( \prec_{\text{cx}} \)-monotone;
   
   (v) \( GS^\lambda_p \) is a coherent risk measure;
   
   (vi) \( \lambda \in [0, 1/2] \).
Gini Shortfall

- Unlike other distortion risk measures, a Gini Shortfall has a simple non-parametric estimator

\[
\widehat{GS}_p^\lambda = \frac{1}{m} \sum_{i=1}^{m} X_i + \frac{\lambda}{m(m-1)} \sum_{i,j=1}^{m} |X_i - X_j|,
\]

where \(X_1, \ldots, X_m\) are the largest \(m = \lfloor np \rfloor\) observations in an iid sample of size \(n\).

- A Gini shortfall is well defined on \(L^1\) and is continuous with respect to \(L^1\) convergence.
Gini Shortfall

Figure: $\text{ES}_p(X)$ and $\text{TGini}_p(X) = \mathbb{E}[|X_p^* - X_p^{**}|]$, $p \in [0.9, 0.99]$ for skew-t risks with $\alpha = 2$ and $\nu = 2$ (left) and $\alpha = 2$ and $\nu = 1.2$ (right)
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Elicitability

Quoting Acerbi-Szekely 2014:

“Eliciwhat?”

*Risk professionals had never heard of elicitability until 2011, when Gneiting proved that ES is not elicitable as opposed to VaR. This result sparked a confusing debate.*

- In 2011, a notion is proposed for comparing risk measure forecasts: *elicitability* (Gneiting 2011)
- Issues related to elicitability soon raised the attention of regulators in the Bank for International Settlements (mentioned in their official consultative document in May 2012)

(earlier study in statistical literature: Osband 1985, Lambert-Pennock-Shoham 2008)
Elicitability

Modified definition of elicitability for law-invariant risk measures

Definition 6
A law-invariant risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is elicitable if there exists a function $S : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho(X) = \min \left\{ \arg \min_{x \in \mathbb{R}} \mathbb{E}[S(x, X)] \right\}, \quad X \in \mathcal{X}.$$ 

- Typically one may further require $S(x, y) \geq 0$ and $S(x, x) = 0$ for $x, y \in \mathbb{R}$.
Elicitability

Assuming all integrals are finite:

- the mean is elicitable with

  \[ S(x, y) = (x - y)^2. \]

- the median is elicitable with

  \[ S(x, y) = |x - y|. \]

- \( \text{VaR}_p \) is elicitable with

  \[ S(x, y) = (1 - p)(x - y)_+ + p(y - x)_+. \]

- an expectile\(^2\) \( e_p \) is elicitable with

  \[ S(x, y) = (1 - p)(x - y)_+^2 + p(y - x)_+^2. \]

\(^2\)introduced by Newey-Powell 1987
Comparative forecasting

- for simplicity, suppose that observations are iid
- for a risk measure $\rho$, different forecasting procedures $\rho^{(1)}, \ldots, \rho^{(k)}$
- at time $t-1$, the estimated/modeled value of $\rho(X_t)$ is $\rho^{(i)}_t$
- collect the statistics $S(\rho^{(i)}_t, X_t)$; a summary statistic can typically be chosen as $T_n(\rho^{(i)}) = \frac{1}{n}\sum_{t=1}^{n} S(\rho^{(i)}_t, X_t)$
- the above procedure is model-independent
- forecasting comparison: compare $T_n(\rho^{(1)}), \ldots, T_n(\rho^{(k)})$
  - risk analyst: compare forecasting procedures/models
  - regulator: compare internal model forecasts with a standard model

Estimation procedures of an elicitable risk measure are straightforward to compare.
Elicitable risk measures

Some characterizations (under suitable conditions)

- (Gneiting 2011)
  VaR is elicitable whereas ES is not.

- (Ziegel 2016)
  Among all coherent risk measures, only expectiles (including the mean) are elicitable.

- (Bellini-Bignozzi 2015, Delbaen-Bellini-Bignozzi-Ziegel 2016)
  Among all convex risk measures, only shortfall risk measures are elicitable.

- (Kou-Peng 2016, W.-Ziegel 2015)
  Among all distortion risk measures, only the mean and the quantiles are elicitable.

- (Acerbi-Székely 2014, Fissler-Ziegel 2016)
  (VaR,ES) is co-elicitable
Elicitable risk measures

Remarks.

- For a $p$-tail pair $(\rho, \rho^*)$: elicitability cannot be pass from $\rho^*$ to $\rho$: take $(\text{ES}_p, E)$. $E$ is elicitable, whereas $\text{ES}_p$ is not.

- A necessary condition for elicitability is closely related to the class of shortfall risk measure (a result of Weber 2006).
Elicitable tail risk measures

Implications.

For monetary risk measures:

\[
\text{Shortfall risk measure} \implies \text{Elicitability}
\]

\[
\uparrow \quad (\text{Weber 2006}) \quad \downarrow (\text{Osband 1985})
\]

CxLS + DLC + AP \quad \implies \quad CxLS

Additional assumptions.

(A8) *Distribution-wise lower-semi-continuous* (DLC). \( \liminf_{n \to \infty} \rho(X_n) \geq \rho(X) \) for \( X_n \to X \) in distribution as \( n \to \infty \).

(A9) *Absorbing property* (AP). There exists \( x \in \mathbb{R} \) such that for all \( y \in \mathbb{R} \), \( Z_\lambda \in \mathcal{A}_\rho \) where \( Z_\lambda \sim (1 - \lambda)\delta_x + \lambda\delta_y \in \mathcal{N}_\rho \) for some \( \lambda > 0 \).
Elicitable tail risk measures

Theorem 7

1. For any \( p \in (0, 1) \), the only elicitable \( p \)-tail convex risk measure is \( \text{VaR}_{1}^{L} \) (the essential supremum).

2. For \( p \in (0, 1) \), a monetary and positively homogeneous \( p \)-tail risk measure \( \rho \) satisfying DLC is elicitable if and only if \( \rho = \text{VaR}_{q}^{L} \) for some \( q \in (p, 1] \).

Remark.

- A new axiomatic characterization of VaRs
- To arrive at \( \text{VaR}_{q}^{R} \): “lower-continuity” \( \rightarrow \) “upper-continuity”, and “min” \( \rightarrow \) “max” in the definition of elicitable risk measures
- Full characterization of elicitable \( p \)-tail risk measures is available
Question.

Suppose that $(\rho, \rho^*)$ is a $p$-tail pair of risk measures.

- $\rho^*$ is elicitable $\Rightarrow$ $(\text{VaR}_p^R, \rho)$ is co-elicitable?
- $(\text{VaR}_p^R, \rho)$ is co-elicitable $\Rightarrow$ $\rho^*$ is elicitable?

Note that $(\text{VaR}_p^R, \text{ES}_p)$ is co-elicitable under suitable continuity conditions.
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Risk aggregation

**Target:** for given univariate distributions $F_1, \ldots, F_n$, calculate

$$\sup\{\rho(S) : S \in S_n(F_1, \ldots, F_n)\}$$

where $S_n(F_1, \ldots, F_n)$ is the aggregation set defined as

$$S_n(F_1, \ldots, F_n) = \{X_1 + \cdots + X_n : X_i \in \mathcal{X}, X_i \sim F_i, \ i = 1, \ldots, n\}.$$

- This setting is called risk aggregation with dependence uncertainty
- A particularly relevant case is $\rho = \text{VaR}_p^L$ or $\rho = \text{VaR}_p^R$ for some $p \in (0, 1)$.

(e.g. Embrechts-Puccetti-R{"u}schendorf 2013, W.-Peng-Yang 2013, Embrechts-Wang-W. 2015)
Risk aggregation

For $X \in L^0$, $F_X^{[p]}$ is the distribution of $X_p$.

**Theorem 8**

Let $p \in (0, 1)$, $(\rho, \rho^*)$ be a $p$-tail pair of monotone risk measures. For any univariate distributions $F_1, \ldots, F_n$, we have

$$\sup\{\rho(S) : S \in S_n(F_1, \ldots, F_n)\} = \sup\{\rho^*(T) : T \in S_n(F_1^{[p]}, \ldots, F_n^{[p]})\}.$$ 

**Remark.**

- monotonicity is essential for the above equation to hold
Risk aggregation

For the cases of VaR and ES:

(i) Take $\rho = \text{VaR}^R_p$.

$$\sup\{\text{VaR}^R_p(S) : S \in S_n(F_1, \ldots, F_n)\}$$

$$= \sup\{\text{ess-inf}(T) : T \in S_n(F_1^{[p]}, \ldots, F_n^{[p]})\}.$$ 

(ii) Take $\rho = \text{ES}_p$. For any $X_1, \ldots, X_n \in L^1$,

$$\text{ES}_p(X_1 + \cdots + X_n) \leq \sup\{\mathbb{E}[T] : T \in S_n(F_{X_1}^{[p]}, \ldots, F_{X_n}^{[p]})\}$$

$$= \sum_{i=1}^n \mathbb{E}[(X_i)_p] = \sum_{i=1}^n \text{ES}_p(X_i),$$

which is (yet another proof of) the classic subadditivity of $\text{ES}_p$.

Dual representations

Suppose \( p \in (0, 1) \), \( \mathcal{D}_p \) is the set of distribution functions on \( [p, 1] \) and \( \rho \) is a functional mapping \( \mathcal{X} = L^\infty \) to \( \mathbb{R} \).

(i) \( \rho \) is a comonotonically additive and coherent \( p \)-tail risk measure if and only if there exists \( g \in \mathcal{D}_p \) such that

\[
\rho(X) = \int_p^1 \mathbb{E}S_q(X) dg(q), \quad X \in \mathcal{X}.
\]

(ii) \( \rho \) is a coherent \( p \)-tail risk measure if and only if there exists a set \( \mathcal{G} \subset \mathcal{D}_p \) such that

\[
\rho(X) = \sup_{g \in \mathcal{G}} \int_p^1 \mathbb{E}S_q(X) dg(q), \quad X \in \mathcal{X}.
\]

(i) and (ii) on non-tail risk measures: Kusuoka 2001, Jouini-Schachermayer-Touzi 2006.
Dual representations

(iii) $\rho$ is a convex and monetary $p$-tail risk measure if and only if there exists a function $\nu : D_p \to \mathbb{R}$ such that

$$\rho(X) = \sup_{g \in D_p} \left\{ \int_p^1 ES_q(X)dg(q) - \nu(g) \right\}, \ X \in \mathcal{X}.$$

(iv) $\rho$ is a $\prec_{cx}$-monotone and monetary $p$-tail risk measure if and only if there exists a set $\mathcal{H}$ of functions mapping $[p, 1]$ to $\mathbb{R}$ such that

$$\rho(X) = \inf_{\alpha \in \mathcal{H}} \sup_{q \in [p, 1]} \{ ES_q(X) - \alpha(q) \}, \ X \in \mathcal{X}.$$

(iii) and (iv) on non-tail risk measures: Frittelli-Rosazza Gianin 2005, Mao-W. 2016

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Progress of the talk

Motivation

Measures of tail risk

Gini Shortfall

Elicitable tail risk measures

Risk aggregation and dual representations

Conclusion

References
Concluding remarks

- replacing a generic risk measure by its tail counterpart is philosophically analogous to replacing the expectation by an ES
- Gini shortfall seems to be a promising risk measure to consider
- potential applications and future research in portfolio selection, decision analysis, risk sharing, etc.
- better position VaR and ES among all tail risk measures
- generalization of the tail distributional transform \( X \mapsto X_p \)
  - choose a general distributional transform \( T : \mathcal{X} \to \mathcal{X} \) and a law-invariant risk measure \( \rho^* \)
  - define a risk measure \( \rho_T(X) = \rho^*(T(X)), X \in \mathcal{X} \).
  - study properties of the triplet \( (\rho_T, \rho^*, T) \)
Reference I


Reference II


Reference IV


Thank you

The talk is based on

- Liu, F. and Wang, R. (2016)
  A theory for measures of tail risk.
  Manuscript available at https://ssrn.com/abstract=2841909

  Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks.
  *Journal of Banking and Finance*, forthcoming.
  Manuscript available at https://ssrn.com/abstract=2836281