Sum of Two Standard Uniform Random Variables

Ruodu Wang
http://sas.uwaterloo.ca/~wang

Department of Statistics and Actuarial Science
University of Waterloo, Canada

Dependence Modeling in Finance, Insurance and Environmental Science
Munich, Germany May 17, 2016

Based on joint work with Bin Wang (Beijing)
In this talk we discuss this problem:

\[ X_1 \sim U[-1, 1], \ X_2 \sim U[-1, 1] \]

what is a distribution (cdf) of \( X_1 + X_2 \)?

A difficult problem with no applications (?)
In an atomless probability space:

- $F_1, \ldots, F_n$ are $n$ distributions
- $X_i \sim F_i$, $i = 1, \ldots, n$
- $S_n = X_1 + \cdots + X_n$

Denote the set of possible aggregate distributions

$$D_n = D_n(F_1, \cdots, F_n) = \{ \text{cdf of } S_n | X_i \sim F_i, \ i = 1, \cdots, n \}.$$

Primary question: Characterization of $D_n$.

- $D_n$ is non-empty, convex, and closed w.r.t. weak convergence
Generic Formulation

For example:

- $X_i$: individual risks; $S_n$: risk aggregation
- fixed marginal; unknown copula

Classic setup in Quantitative Risk Management

- Secondary question: what is $\sup_{F \in D_n} \rho(F)$ for some functional $\rho$ (risk measure, utility, moments, ...)?
- Risk aggregation with dependence uncertainty, an active field over the past few years:
  - Embrechts et. al. (2014 Risks) and the references therein
  - 20+ papers in the past 3 years
Assume that $F_1, \ldots, F_n$ have finite means $\mu_1, \ldots, \mu_n$, respectively.

- Necessary conditions:
  - $S_n \prec_{cx} F_1^{-1}(U) + \cdots + F_n^{-1}(U)$
  - In particular, $\mathbb{E}[S_n] = \mu_1 + \cdots + \mu_n$
  - Range($S_n$) $\subset \sum_{i=1}^n$ Range($X_i$)

- Suppose $\mathbb{E}[T] = \mu_1 + \cdots + \mu_n$. Then

  $$F_T \in D_n(F_1, \ldots, F_n) \iff (F_{-T}, F_1, \ldots, F_n) \text{ is jointly mixable}$$

For a theory of joint mixability

Some Observations

- Joint mixability is an open research area
- A general analytical characterization of $\mathcal{D}_n$ or joint mixability is far away from being clear
- We tune down and look at standard uniform distributions and $n = 2$
Simple Examples

\[ X_1 \sim U[-1, 1], \ X_2 \sim U[-1, 1], \ S_2 = X_1 + X_2. \]

Obvious constraints

- \( \mathbb{E}[S_2] = 0 \)
- range of \( S_2 \) in \([-2, 2]\)
- \( \text{Var}(S_2) \leq 4/3 \)
- \( S_2 \preceq_{cx} 2X_1 \) (sufficient?)

Ruodu Wang  (wang@uwaterloo.ca)  Sum of two uniform random variables
Simple Examples

Are the following distributions possible for $S_2$?

- Triangular distribution
- Uniform distribution
- Dirac delta function
- Uniform distribution
- Uniform distribution
- Uniform distribution
Simple Examples: More...

A Small Copula Game...

\[
\mathbb{P}(S_2 = -4/5) = 1/2, \quad \mathbb{P}(S_2 = 4/5) = 1/2
\]
Progress of the Talk

1. Question
2. Some Examples
3. Some Answers
4. Some More
5. References
Existing Results

Let $\mathcal{D}_2 = \mathcal{D}_2(U[-1, 1], U[-1, 1])$. Below are implied by results in Wang-W. (2016 MOR)

- Let $F$ be any distribution with a **monotone** density function. Then $F \in \mathcal{D}_2$ if and only if $F$ is supported in $[-2, 2]$ and has zero mean.

- Let $F$ be any distribution with a **unimodal and symmetric** density function. Then $F \in \mathcal{D}_2$ if and only if $F$ is supported in $[-2, 2]$ and has zero mean.

- $U[-a, a] \in \mathcal{D}_2$ if and only if $a \in [0, 2]$ (a special case of both).
  - The case $U[-1, 1] \in \mathcal{D}_2$ is given in Rüschendorf (1982 JAP).
Unimodal Densities

A natural candidate to investigate is the class of distributions with a unimodal density.

**Theorem 1**

Let $F$ be a distribution with a unimodal density on $[-2, 2]$ and zero mean. Then $F \in \mathcal{D}_2$.

- Both the two previous results are special cases
- For bimodal densities we do not have anything concrete
A second candidate is a distribution which dominates a portion of a uniform distribution.

**Theorem 2**

Let $F$ be a distribution supported in $[a - b, a]$ with zero mean and density function $f$. If there exists $h > 0$ such that $f \geq \frac{3b}{4h}$ on $[-h/2, h/2]$, then $F \in D_2$.

- The density of $F$ dominates $3b/4$ times that of $U[-h/2, h/2]$
Continuous distributions seem to be a dead end; what about discrete distributions? Let us start with the simplest cases.
Theorem 3

Let $F$ be a bi-atomic distribution with zero mean supported on \( \{a - b, a\} \). Then $F \in \mathcal{D}_2$ if and only if $2/b \in \mathbb{N}$.

For given $b > a > 0$, there is only one distribution on \( \{a - b, a\} \) with mean zero.
Tri-atomic Distributions

For a tri-atomic distribution $F$, write $F = (f_1, f_2, f_3)$ where $f_1, f_2, f_3$ are the probability masses of $F$

- On given three points, the set of tri-atomic distributions with mean zero has one degree of freedom.
- We study the case of $F$ having an “equidistant support” $\{a - 2b, a - b, a\}$.

For $x > 0$, define a “measure of non-integrity”

$$\lceil x \rceil = \min \left\{ \frac{\lceil x \rceil}{x} - 1, 1 - \frac{\lceil x \rceil}{x} \right\} \in [0, 1].$$

Obviously $\lceil x \rceil = 0 \iff x \in \mathbb{N}$. 
Tri-atomic Distributions

**Theorem 4**

Suppose that \( F = (f_1, f_2, f_3) \) is a tri-atomic distribution with zero mean supported in \( \{a - 2b, a - b, a\} \), \( \epsilon > 0 \) and \( a \leq b \). Then \( F \in \mathcal{D}_2 \) if and only if it is the following three cases.

(i) \( a = b \) and \( f_2 \geq \lceil \frac{1}{b} \rceil \).

(ii) \( a < b \) and \( \frac{1}{b} \in \mathbb{N} \).

(iii) \( a < b, \frac{1}{b} - \frac{1}{2} \in \mathbb{N} \) and \( f_2 \geq \frac{a}{2} \).

\[ \text{cf. Theorem 3 (condition } 2/b \in \mathbb{N} \text{)} \]
The corresponding distributions in Theorem 4:

(i) \((f_1, f_2, f_3) \in \text{cx}\{(0, 1, 0), \frac{1}{2}(1 - \lceil \frac{1}{b} \rceil, 2\lceil \frac{1}{b} \rceil, 1 - \lceil \frac{1}{b} \rceil)\}\).

(ii) \((f_1, f_2, f_3) \in \text{cx}\{(0, \frac{a}{b}, 1 - \frac{a}{b}), \frac{1}{2}(\frac{a}{b}, 0, 2 - \frac{a}{b})\}\).

(iii) \((f_1, f_2, f_3) \in \text{cx}\{(0, \frac{a}{b}, 1 - \frac{a}{b}), \frac{1}{2}(\frac{a}{b} - \frac{a}{2}, a, 2 - \frac{a}{b} - \frac{a}{2})\}\).
Progress of the Talk

1. Question
2. Some Examples
3. Some Answers
4. Some More
5. References
Some More to Expect

- It is possible to further characterize $n$-atomic distributions with an equidistant support (things get ugly though).
- We guess: for any distribution $F$
  - with an equidistant support, or
  - with finite density and a bounded support,

there exists a number $M > 0$ such that

$$F \in \mathcal{D}_2(\mathbb{U}[-m, m], \mathbb{U}[-m, m]) \text{ for all } m \in \mathbb{N} \text{ and } m > M.$$
Some More to Think

- Two uniforms with different lengths?
- Three or more uniform distributions?
- Other types of distributions?
- Applications?

We yet know very little about the problem of $D_2$


Thank you for your kind attention