Risk Aversion in Regulatory Capital Principles

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Outline

1. Regulatory capital principles
2. Risk measures in financial decisions: an example
3. Consistent risk measures
4. Mathematical Characterization
5. Risk sharing
6. Discussions
7. References

Based on joint work with Tiantian Mao (USTC, China)
Risk measures as regulatory capital principles

A (regulatory) risk measure is a functional \( \rho : \mathcal{X} \to (-\infty, \infty] \) which calculates the amount of regulatory capital of a financial institution taking a risk (random loss) \( X \) in a fixed period.

- \( (\Omega, \mathcal{F}, \mathbb{P}) \) is an atomless probability space
- \( \mathcal{X} \) is a convex cone of random variables
  - e.g. \( \mathcal{X} = L^q(\Omega, \mathcal{F}, \mathbb{P}), \; q \in [1, \infty] \)
- \( X \in \mathcal{X} \) represent loss/profit (discounted to present)

Very general question

What is a good risk measure to use?
### Regulatory Capital Principles

<table>
<thead>
<tr>
<th></th>
<th>regulator</th>
<th>firm manager</th>
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<tbody>
<tr>
<td><strong>usage</strong></td>
<td>external regulation</td>
<td>internal management performance analysis</td>
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<td>capital allocation</td>
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<td><strong>interest</strong></td>
<td>social welfare</td>
<td>shareholders</td>
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<td><strong>risk</strong></td>
<td>systemic risk</td>
<td>risk of a single firm</td>
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<tr>
<td><strong>role</strong></td>
<td>designs a principle</td>
<td>reacts to a principle</td>
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<tr>
<td><strong>goal</strong></td>
<td>maintain enough capital</td>
<td>reduce regulatory capital</td>
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<tr>
<td><strong>risk-averse</strong></td>
<td>yes</td>
<td>not necessarily</td>
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Value-at-Risk and Expected Shortfall

**Value-at-Risk (VaR) at level** $p \in (0, 1)$

$$\text{VaR}_p : L^0 \to \mathbb{R},$$

$$\text{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}.$$

**Expected Shortfall (ES/TVaR/CVaR/AVaR) at level** $p \in (0, 1)$

$$\text{ES}_\beta : L^1 \to \mathbb{R},$$

$$\text{ES}_p(X) = \frac{1}{1 - p} \int_p^1 \text{VaR}_q(X) dq, \quad p \in (0, 1).$$
The ongoing debate on “VaR versus ES”:

- Basel III (mixed; in transition from VaR to ES as standard metric for market risk\(^1\))
- Solvency II (VaR based)
- Swiss Solvency Test (ES based)

\(^1\)e.g. **Basel Committee on Banking Supervision**: Standards, January 2016, Minimum capital requirements for Market Risk.
Value-at-Risk and Expected Shortfall

Many perspectives

- regulator’s versus firms’ standpoints
- economic interpretation
- statistical issues: estimation, robustness, backtesting, model uncertainty
- computation, simulation and optimization
- systemic risk

There is no single “perfect” risk measure

Some academic references

- Embrechts et al. (2014)
We provide a new perspective: incorporating risk aversion to the above issue on risk measures.
Standard Properties of Risk Measures

Some standard properties of risk measures

(M) Monotonicity: $\rho(X) \leq \rho(Y)$ for $X, Y \in \mathcal{X}$, $X \leq Y$ almost surely;

(TI) Translation-invariance: $\rho(X - m) = \rho(X) - m$ for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$.

(LI) Law-invariance: $\rho(X) = \rho(Y)$ if $X, Y \in \mathcal{X}$ and $X \overset{d}{=} Y$.

Definition 1

A monetary risk measure is a functional on $\mathcal{X}$ satisfying (M) and (TI).

- VaR and ES are monetary and law-invariant.
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A simplified example:

- Ω = {ω₁, ω₂, ω₃}: future (e.g. one-year) economic states
  - ω₁: a normal economic state
  - ω₂: an adverse economic state
  - ω₃: an extreme scenario
- P(ω₁) = 0.99, P(ω₂) = 0.0099 and P(ω₃) = 0.0001
- A financial institution has to choose between two risks (decisions)
Simple Example

Risks $X$ and $Y$ (in millions of USD):

$$X = \begin{cases} 
-1 & \omega = \omega_1, \\
10 & \omega = \omega_2, \\
20 & \omega = \omega_3,
\end{cases} \quad Y = \begin{cases} 
-1.1 & \omega = \omega_1, \\
9.9 & \omega = \omega_2, \\
2,000 & \omega = \omega_3.
\end{cases}$$

Possible interpretations:

- $X$ is benchmark - $Y$ is $X$ plus an bet against event $\omega_3$
  (e.g. AAA bond with high leverage)
- $Y$ is benchmark - $X$ is $Y$ plus a hedge against event $\omega_3$
  (e.g. insurance contract)

$\mathbb{P}(Y < X) = 99.99\%$
Simple Example

- Assume that the financial institution has 10M (economic) capital
  - \( \text{VaR}_{0.999}(X) = 10, \text{VaR}_{0.999}(Y) = 9.9 \)

- Which risk would the financial institution prefer?
  - The manager of the financial institution is not necessarily risk averse
  - Limited liability
  - \( \mathbb{P}(\omega_3) \) is too small to notice or accurately model

- Which risk would a regulator prefer?
  - A regulator cares about loss to the society
  - What if all firms in the system are doing this? ... Aggregation!
Question

How can the regulator leads/encourages the financial institution to choose $X$ over $Y$?

Idea:

(1) A firm has incentives to reduce its regulatory capital
   - Firms are “effectively risk averse” because holding capital is costly

(2) View a regulatory risk measure $\rho$ as a decision principle for the firm

(3) Choose a properly designed $\rho$
A regulator uses $\rho$ to calculate regulatory capital

- Formally, assume that for two decisions $X$ and $Y$, if $\rho(X) \ll \rho(Y)$, then a firm has the incentive to choose $X$ (smaller capital) over $Y$ (larger capital).
- If the regulator prefers $X$ to $Y$, then she should design $\rho$ such that $\rho(X) < \rho(Y)$.
- In the previous example

<table>
<thead>
<tr>
<th></th>
<th>$X$</th>
<th>$Y$</th>
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<tbody>
<tr>
<td>VaR$_{0.999}$</td>
<td>10</td>
<td>9.9</td>
</tr>
<tr>
<td>ES$_{0.999}$</td>
<td>11</td>
<td>208.91</td>
</tr>
<tr>
<td>StDev</td>
<td>1.109</td>
<td>20.039</td>
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</tbody>
</table>
Financial Decisions and Risk Preference

What is a suitable preference for the regulator?

- very complicated question
- for the interest of the society
- decision theory $\leftrightarrow$ regulatory risk measures
## Progress of the Talk

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3. **Consistent risk measures**
4. Mathematical Characterization
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Expected Loss to the Society

A company has capital $K$ and decides between two risks $X, Y \in \mathcal{X} \subset L^1$.

- If $\mathbb{E}[(X - K)_+] \leq \mathbb{E}[(Y - K)_+]$ then taking $X$ has less expected loss to the society.

- If $\mathbb{E}[(X - K)_+] \leq \mathbb{E}[(Y - K)_+]$ holds for all $K$, then it is reasonable that $X$ requires a smaller capital.

Formally, define the property

**(EL)** Consistency with expected loss to the society: for $X, Y \in \mathcal{X}$,

$$\rho(X) \leq \rho(Y) \text{ if } \mathbb{E}[(X - K)_+] \leq \mathbb{E}[(Y - K)_+] \text{ for all } K \in \mathbb{R}.$$ 

**(EL)** is equivalent to the consistency with respect to second-order stochastic dominance (SSD).
Definition 2 (Second-order stochastic dominance)

For $X, Y \in L^1$, $X$ has second-order stochastic dominance (SSD) over $Y$, denoted as $X \prec_{sd} Y$, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing convex functions $f$ such that the expectations exist.

- Also known as increasing convex order or stop-loss order
- $X \prec_{sd} Y$ in the previous three-state example

(SC) SSD consistency: $\rho(X) \leq \rho(Y)$ if $X \prec_{sd} Y$, $X, Y \in \mathcal{X}$.

- (SC) is called strong risk aversion in decision theory
- (SC) $\iff$ (EL)
Assume $\mathcal{X} \subset L^1$ in the following.

**Definition 3 (Consistent risk measures)**

A risk measure is a **consistent risk measure** if it satisfies (SC) and (TI).

- Consistent risk measures are monetary
- Interpretation: the regulator penalizes more risky financial decisions (ones that have higher expected social impact)
Consistent Risk Measures

Some examples

- An Expected Shortfall $\text{ES}_p$, $p \in (0, 1)$ is consistent
- The mean $\mathbb{E}[\cdot]$ on $L^1$ is consistent
- Any law-invariant convex risk measure on $L^\infty$ is consistent
- Any finite law-invariant convex risk measure on $L^q$, $q \geq 1$ is consistent
- Any Value-at-Risk $\text{VaR}_p$, $p \in (0, 1)$ is not consistent

Is a consistent risk measure necessarily convex?
Properties

Similar properties for risk measures

(CC) Convex order consistency: \( \rho(X) \leq \rho(Y) \) if \( X \prec_{\text{cx}} Y \), \( X, Y \in \mathcal{X} \).

(DM) Dilatation monotonicity: \( \rho(X) \leq \rho(Y) \) if \( (X, Y) \in \mathcal{X}^2 \) is a martingale.

(DC) Diversification consistency: \( \rho(X + Y) \leq \rho(X^c + Y^c) \) if \( X, Y, X^c, Y^c \in \mathcal{X} \), \( X \overset{d}{=} X^c \), \( Y \overset{d}{=} Y^c \), and \( (X^c, Y^c) \) is comonotonic.
Proposition 4

For a monetary risk measure on $L^\infty$, (SC), (EL), (CC), (DM), (DC) are equivalent. Moreover, each of them implies (LI).
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The next question is a characterization of all consistent risk measures.

- We assume $\mathcal{X} = L^\infty$ for simplicity
- All results hold for $\mathcal{X} = L^q$, $q \geq 1$
Characterization Theorem

**Theorem 5**

A risk measure $\rho$ on $L^{\infty}$ is consistent if and only if there exists a set $\mathcal{G}$ of functions mapping $(0,1)$ to $(-\infty, \infty]$ such that

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{p \in (0,1)} \left\{ ES_p(X) - g(p) \right\}, \quad X \in L^{\infty}. \quad (1)$$

- Example: If $\rho$ is $ES_p \ (p \in (0,1))$, then one can take $\mathcal{G} = \{g_p\}$ where $g_p(p) = 0$ and $g_p(x) = \infty$ for $x \in (0,1) \setminus p$.

- $\mathcal{G}$ in (1) is not unique. It may be chosen as the adjustment set of $\rho$

$$\mathcal{G} = \{g_Y : \ Y \in \mathcal{X}, \ \rho(Y) \leq 0\},$$

where $g_Y : (0,1) \to \mathbb{R}, \ p \mapsto ES_p(Y)$. 

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Characterization Theorem

On the representation:

\[ \rho(X) = \inf_{g \in G} \sup_{p \in (0,1)} \{ \text{ES}_p(X) - g(p) \}, \quad X \in L^\infty. \]

- \( g \in G \) are benchmarks: if for some \( g \in G \), \( \text{ES}(X) \leq g(\cdot) \), then \( \rho(X) \leq 0 \) (an accepted risk without extra capital); otherwise \( \rho(X) > 0 \) (or \( \geq 0 \)).

- Any risk-averse regulator or risk manager is essentially using a collection of Expected Shortfalls up to some adjustments.
Relation to Classic Risk Measures

Classic properties in the theory of monetary risk measures

**(PH)** Positive homogeneity: \( \rho(\lambda X) = \lambda \rho(X) \) for all \( \lambda \in (0, \infty) \) and \( X \in \mathcal{X} \);

**(CX)** Convexity: \( \rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y) \) for all \( \lambda \in [0, 1] \) and \( X, Y \in \mathcal{X} \);

**(CA)** Comonotonic additivity: \( \rho(X + Y) = \rho(X) + \rho(Y) \) if \( (X, Y) \in \mathcal{X}^2 \) is comonotonic.

**Definition 6**

A risk measure is called a **convex risk measure** if it satisfies (M), (TI) and (CX). A risk measure is called a **coherent risk measure** if it satisfies (M), (TI), (PH) and (CX).

Consistent risk measures are closely related to law-invariant convex risk measures.

**Theorem 7**

A risk measure $\rho$ on $L^\infty$ is consistent if and only if there exists a set $C$ of law-invariant convex risk measures such that

$$\rho(X) = \inf_{\tau \in C} \tau(X), \quad X \in L^\infty.$$
Yet we obtain a new characterization of convex (coherent) risk measures.

**Proposition 8**

A law-invariant risk measure $\rho$ on $L^\infty$ is a convex (resp. coherent) risk measure if and only if there exists a convex set (resp. convex cone) $\mathcal{G}$ of functions mapping $(0,1)$ to $(-\infty,\infty]$ such that

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{p \in (0,1)} \{ \text{ES}_p(X) - g(p) \}, \quad X \in L^\infty.$$
Consistency vs Convexity

Consistency versus convexity:

(SC) Consistency: \( \rho(X) \leq \rho(Y) \) if \( X \prec_{sd} Y, X, Y \in \mathcal{X} \).

(CX) Convexity: \( \rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y) \) for all \( \lambda \in [0, 1] \) and \( X, Y \in \mathcal{X} \).

(i) Consistency compares between risks (decisions) while convexity does not

(ii) For risk-types other than market risk, portfolio diversification is not appropriate

(iii) There is no direct reason why a regulator would favour diversification in a single company, unless some social benefit could be expected (cf. Ibragimov-Jaffee-Walden 2011)
Kusuoka Representations

Let $\mathcal{P}$ be the set of all probability measures on $[0, 1]$ and $\mathcal{U}$ be the set of all functions mapping $\mathcal{P}$ to $\mathbb{R}$.

A law-invariant coherent risk measure $\rho$ on $L^\infty$ has the following representation

$$\rho = \sup_{h \in \mathcal{R}} \left\{ \int_0^1 ES_p h(p) \right\} \quad \text{for some } \mathcal{R} \subset \mathcal{P}. $$

A law-invariant convex risk measure $\rho$ on $L^\infty$ has the following representation

$$\rho = \sup_{h \in \mathcal{P}} \left\{ \int_0^1 ES_p h(p) - \alpha(h) \right\} \quad \text{for some } \alpha \in \mathcal{U}. $$

(Kusuoka 2001, Frittelli-Rosazza Gianin 2005)
Kusuoka Representations

Grand summary: for a risk measure on $L^\infty$, 

$$(\text{TI})+(\text{SC}) = \inf_{\alpha \in \mathcal{V}} \sup_{h \in \mathcal{P}} \left\{ \int_0^1 ES_p \, dh(p) - \alpha(h) \right\} \quad \text{for some } \mathcal{V} \subset \mathcal{U}$$

$$(\text{CX}) \quad \sup_{h \in \mathcal{P}} \left\{ \int_0^1 ES_p \, dh(p) - \alpha(h) \right\} \quad \text{for some } \alpha \in \mathcal{U}$$

$$(\text{PH}) \quad \sup_{h \in \mathcal{R}} \left\{ \int_0^1 ES_p \, dh(p) \right\} \quad \text{for some } \mathcal{R} \subset \mathcal{P}$$

$$(\text{CA}) \quad \int_0^1 ES_p \, dh(p) \quad \text{for some } h \in \mathcal{P}.$$

Remark: $(\text{TI})+(\text{SC})+(\text{CA})$ is sufficient for the last representation.
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Risk Sharing

General setup

- $n$ agents sharing a total risk $X \in \mathcal{X}$
- $\rho_1, \ldots, \rho_n$: underlying risk measures

Target: for $X \in \mathcal{X}$, find a Pareto-optimal solution of $X$ to minimize

$$\rho_1(X_1), \ldots, \rho_n(X_n)$$

over the set of all allocations:

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$
Theorem 9

Suppose that $\rho_1, \ldots, \rho_n$ are consistent risk measures on $\mathcal{X} = L^q$, $q \in [1, \infty]$ with adjustment sets $G_1, \ldots, G_n$, respectively. An allocation $(X_1, \ldots, X_n) \in A_n(X)$ is Pareto-optimal if and only if

$$\sum_{i=1}^{n} \rho_i(X_i) = \rho^*(X),$$

where $\rho^*$ is a consistent risk measure with adjustment set $\sum_{i=1}^{n} G_i$.

In particular,

$$\rho^*(X) = \inf_{g \in G_1 + \cdots + G_n} \sup_{\alpha \in [0,1]} \{ \text{ES}_\alpha(X) - g(\alpha) \}, \quad X \in \mathcal{X}.$$
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Suitable risk measures for regulation

On the current debates regarding the desirability of VaR and ES:

- A suitable risk measure applied in regulatory practice should encourage **prudent** and **socially responsible** financial decisions
  - Financial institutions are not necessarily risk-averse or socially responsible for their own interest; a regulator should push them towards risk-aversion

- ES is the basis for any consistent risk measure - supporting the transition **from VaR to ES** in the recent Basel documents

- ES is the **only candidate** which **preserves consistency** and also has **simple form** and **clear economic interpretation**
Further remarks:

- Consistency is more *natural* than convexity for a regulator
- One can construct *non-convex* consistent risk measures
  - As far as we are aware of, there are no non-convex consistent risk measures in simple analytical forms other than a minimum
- Criteria for a desirable risk measure used in banking and insurance regulation may vary
- Bring more in decision theory to risk measures and regulation
References


Thank you for your kind attendance

The manuscript can be downloaded at
http://ssrn.com/abstract=2658669