How Much Does the Dependence Structure Matter?

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Outline

Part I - Dependence Uncertainty

Challenges in dependence

- Two aspects of modeling and inference: **marginal distribution** and **dependence structure**. *"copula thinking"*

Margins vs Dependence

Part I - Dependence Uncertainty

Challenges in dependence

- Two aspects of modeling and inference: **marginal distribution** and **dependence structure**. *"copula thinking"*

Margins vs Dependence

A practical setup: known margins, unknown dependence.

Distribution of the sum

Let

$$S_n = X_1 + \cdots + X_n$$

where $X_i \sim F_i$, F_i is fixed, $i = 1, \ldots, n$, and the joint distribution of $(X_1, \ldots, X_n)$ is unknown.

Key question

What is a possible distribution of S_n?

Attention coming from QRM

- Most research looks at extreme values of some quantities (e.g. risk measures) on the aggregate position S_n
  - Book: Rüschendorf (2013)

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- Most research looks at extreme values of some quantities (e.g. risk measures) on the aggregate position S_n
  - Book: Rüschendorf (2013)
- Some recent literature on **risk aggregation and dependence uncertainty**:
  - Dhaene et al. (2012, IME).
  - W., Peng and Yang (2013, F&S)
  - Embrechts, Puccetti and Rüschendorf (2013, JBF)
  - Puccetti and Rüschendorf (2014, J Risk)
  - Puccetti, Wang and W. (2013, IME)
  - Bernard, Jiang and W. (2014, IME)
  - Embrechts et al. (2014, Risks)
  - many preprints

Admissible distributions

Denote the set of admissible distributions

$$\mathfrak{D}_n = \mathfrak{D}_n(F_1, \ldots, F_n) = \{\text{cdf of } S_n | X_i \sim F_i,\ i = 1, \ldots, n\}.$$

Some conclusion:

- $\mathfrak{D}_n$ is a convex set, and closed with respect to weak topology.
- Many open questions on the characterization of $\mathfrak{D}_n$.
- Even finding $\sup_{F \in \mathfrak{D}_n} F(x)$, for fixed $x \in \mathbb{R}$, is generally open.

Admissible distributions

For a fixed n, a relevant notion is joint mixability.

- Notice that

$$F_S \in \mathfrak{D}_n(F_1, \ldots, F_n) \;\Leftrightarrow\; \delta_0 \in \mathfrak{D}_{n+1}(F_1, \ldots, F_n, F_{-S}).$$

- The characterization of joint mixability is equivalent to the characterization of $\mathfrak{D}_n$.

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In this talk, we consider two types of problems in the case of $n \to \infty$.

Part II - Extreme Negative Dependence

A simple setting: assume $X_i \sim F, i \in \mathbb{N}$ and F has mean $\mu < \infty$. We look at the quantity $V_n = \mathrm{Var}(S_n)$.

- What is V_n as $n \to \infty$ if F has finite variance σ^2? Of course it depends on the dependence of the sequence.

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- What is V_n as $n \to \infty$ if F has finite variance σ^2? Of course it depends on the dependence of the sequence.
  - iid case: $V_n = n\sigma^2 = O(n)$.
  - comonotonic case: $V_n = n^2\sigma^2 = O(n^2)$. This gives the largest rate.
  - what about the smallest possible rate of V_n?

Extreme Negative Dependence

Assume $X_i \sim F_i$, $i \in \mathbb{N}$. Two properties:

(a) $S_n - \mathbb{E}[S_n] = O_p(1)$ as $n \to \infty$;

(b) $S_n,\ n \in \mathbb{N}$ have variance bounded by a constant.

Definition 1

We say that a sequence of random variables $(X_i \sim F_i,\ i \in \mathbb{N})$ is *extremely negatively dependent* (END), if **(a)** holds.

Strong Extreme Negative Dependence

Definition 2

We say that $(X_i \sim F_i,\ i \in \mathbb{N})$ is *strongly extremely negatively dependent* (SEND), if **(a)-(b)** hold and

$$\sup_{n\in\mathbb{N}} \mathrm{Var}(S_n) \le \sup_{n\in\mathbb{N}} \mathrm{Var}(Y_1 + \cdots + Y_n)$$

for any sequence of random variables $(Y_i \sim F_i,\ i \in \mathbb{N})$.

Existence of END

An immediate question:

Existence

For given F or $\{F_i, i \in \mathbb{N}\}$, does there exist an (S)END sequence having the corresponding marginal distributions?

Homogeneous margins

Theorem 1 (Wang and W., 2014, preprint)

Suppose F has finite k-th moment, $k \in [1, \infty]$ then there exist a sequence $(X_i \sim F,\ i \in \mathbb{N})$ and $Z \in L^{k-1}$ such that $|S_n - \mathbb{E}[S_n]| \le Z$.

Homogeneous margins

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- **(a)** holds: $(X_i,\ i \in \mathbb{N})$ is END.
- If F has finite third moment, then $V_n = O(1)$, and **(b)** holds. It is SEND in some special cases.
- Proof by construction; explicit form of Z is obtained.
- This result can be easily extended to random vectors.

Inhomogeneous margins

Proposition 2 (Embrechts, Wang and W., 2014, preprint)

Suppose F_i, $i \in \mathbb{N}$ *are supported in a finite interval of length* ℓ*. Then there exists a sequence* $(X_i \sim F_i,\ i \in \mathbb{N})$*, such that* S_n *is supported in a finite interval of length* ℓ*.*

Inhomogeneous margins

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Suppose F_i, $i \in \mathbb{N}$ are supported in a finite interval of length ℓ. Then there exists a sequence $(X_i \sim F_i,\ i \in \mathbb{N})$, such that S_n is supported in a finite interval of length ℓ.

- $(X_i,\ i \in \mathbb{N})$ is END and $V_n = O(1)$.
- For unbounded distributions, we do not have good results yet.
  - A small conjecture: if $\{F_i,\ i \in \mathbb{N}\}$ has uniformly bounded k-th moment, then there exists a sequence $(X_i \sim F_i,\ i \in \mathbb{N})$ and $Z \in L^{k-1}$, such that $|S_n - \mathbb{E}[S_n]| \leq Z$.

Asymptotic equivalence

We denote, for $p \in (0,1)$,

$$\overline{\mathrm{ES}}_p(S_n) = \sup\{\mathrm{ES}_p(S) : S \sim G \in \mathfrak{D}_n(F_1, \ldots, F_n)\},$$

and

$$\overline{\mathrm{VaR}}_p(S_n) = \sup\{\mathrm{VaR}_p(S) : S \sim G \in \mathfrak{D}_n(F_1, \ldots, F_n)\}.$$

- $\overline{\mathrm{ES}}_p(S_n) = \sum_{i=1}^n \mathrm{ES}_p(X_i)$.
- Many papers on $\overline{\mathrm{VaR}}_p(S_n)$; no explicit form in general.

Asymptotic equivalence

Consider the case $n \to \infty$. What happens to $\overline{\mathrm{VaR}}_p(S_n)$?

- Clearly always $\overline{\mathrm{VaR}}_p(S_n) \leq \overline{\mathrm{ES}}_p(S_n)$.

Under some weak conditions,

$$\lim_{n\to\infty} \frac{\overline{\mathrm{ES}}_p(S_n)}{\overline{\mathrm{VaR}}_p(S_n)} = 1.$$

- First form of this equivalence in some homogeneous models found in Puccetti and Rüschendorf (2014, J Risk).
- Full proof available based on END.

Asymptotic equivalence, homogeneous model

Theorem 3

In the homogeneous model, $F_1 = F_2 = \cdots = F$, for $p \in (0,1)$ and $X \sim F$, we have that

$$\lim_{n\to\infty} \frac{1}{n}\overline{\mathrm{VaR}}_p(S_n) = \mathrm{ES}_p(X).$$

- Similar limits hold for a large class of risk measures.

Asymptotic equivalence, general model

Theorem 4 (Embrechts, Wang and W., 2014, preprint)

Suppose the continuous distributions F_i, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $p \in (0,1)$,

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ *is uniformly bounded for some* $k > 1$;

(ii) $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathrm{ES}_p(X_i) > 0.$

Then as $n \to \infty$,

$$\frac{\overline{\mathrm{VaR}}_p(S_n)}{\overline{\mathrm{ES}}_p(S_n)} = 1 - O(n^{-1+1/k}) \to 1.$$

A proof in the homogeneous case

Assume the the marginal distributions F_i are bounded, denoted by F. Let $X \sim F$.

$$\frac{\overline{\mathrm{VaR}}_p(S_n)}{\overline{\mathrm{ES}}_p(S_n)} = \frac{\overline{\mathrm{VaR}}_p(\frac{1}{n}S_n)}{\overline{\mathrm{ES}}_p(\frac{1}{n}S_n)} = \frac{\overline{\mathrm{VaR}}_p(\frac{1}{n}S_n)}{\mathrm{ES}_p(X)}.$$

Now, for $U \sim \mathrm{U}[0,1]$, let $Y \overset{d}{=} F^{-1}(U)|U \le p$, $Y_i \overset{d}{=} F^{-1}(U)|U > p$, $Y_i,\ i \in \mathbb{N}$ are END, and

$$X_i = Y\mathrm{I}_A + Y_i\mathrm{I}_{A^c}$$

where $\mathbb{P}(A) = p$ and A is independent of $Y, Y_i, i \in \mathbb{N}$.

A proof in the homogeneous case II

Then this special choice of S_n has

$$\frac{1}{n}S_n \to Y\mathrm{I}_A + \mathbb{E}[Y_1]\mathrm{I}_{A^c} \quad \text{in } L^\infty,$$

and

$$\frac{\mathrm{VaR}_p(\frac{1}{n}S_n)}{\mathrm{ES}_p(X)} \to \frac{\mathbb{E}[Y_1]}{\mathrm{ES}_p(X)} = 1.$$

Red: comonotonic; blue: END.

- We need END for the L^∞ convergence since VaR is discontinuous with respect to L^1 or a.s. convergence.
- Inhomogeneous case: similar.
- Unbounded distributions: take limit.

Dependence-uncertainty spread

Theorem 5 (Embrechts, Wang and W., 2014)

Take $1 > q \geq p > 0$. *Suppose that the continuous distributions* $F_i,\ i \in \mathbb{N}$, *satisfy (i) and (ii), then*

$$\liminf_{d\to\infty} \frac{\overline{\mathrm{VaR}}_q(S_n) - \underline{\mathrm{VaR}}_q(S_n)}{\overline{\mathrm{ES}}_p(S_n) - \underline{\mathrm{ES}}_p(S_n)} \geq 1.$$

- The uncertainty-spread of VaR is generally bigger than that of ES.
- In recent Consultative Documents of the Basel Committee, $\mathrm{VaR}_{0.99}$ is compared with $\mathrm{ES}_{0.975}$: $p = 0.975$ and $q = 0.99$.

Dependence-uncertainty spread

ES and VaR of $S_n = X_1 + \cdots + X_n$, where

- $X_i \sim \text{Pareto}(2 + 0.1i),\ i = 1, \ldots, 5;$
- $X_i \sim \text{Exp}(i - 5), i = 6, \ldots, 10;$
- $X_i \sim \text{Log–Normal}(0, (0.1(i - 10))^2),\ i = 11, \ldots, 20.$

	$n = 5$			$n = 20$		
	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\text{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\text{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}}{\overline{\text{VaR}}_{0.975}}$		1.08			1.02	

Part III - Convergence Problems

Now let us go back to a previous question.

Question. Suppose $F_1 = F_2 = \mathrm{U}[0,1]$. What is $\mathfrak{D}_2$?

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- The point mass $\delta_1 \in \mathfrak{D}_2$ by taking $X_2 = 1 - X_1 \sim \mathrm{U}[0,1]$;
- $\mathrm{U}[0,2] \in \mathfrak{D}_2$ by taking $X_2 = X_1 \sim \mathrm{U}[0,1]$;

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- The point mass $\delta_1 \in \mathfrak{D}_2$ by taking $X_2 = 1 - X_1 \sim \mathrm{U}[0,1]$;
- $\mathrm{U}[0,2] \in \mathfrak{D}_2$ by taking $X_2 = X_1 \sim \mathrm{U}[0,1]$;
- If $F \in \mathfrak{D}_2$, then it holds that $\delta_1 \prec_{\mathrm{cx}} F \prec_{\mathrm{cx}} \mathrm{U}[0,2]$ where $\prec_{\mathrm{cx}}$ is convex order.
  - Based on classic results on comonotonicity and counter-monotonicity.

Convex lower set

Define a lower set of F w.r.t. convex order:

$$\mathfrak{C}(F) = \{G \in \mathcal{F}^1 : G \prec_{\text{cx}} F\}.$$

where $\mathcal{F}^1$ is the set of distributions with finite mean.

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- $\mathfrak{D}_2 \subset \mathfrak{C}(\mathrm{U}[0,2])$ holds.

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where $\mathcal{F}^1$ is the set of distributions with finite mean.

- $\mathfrak{D}_2 \subset \mathfrak{C}(\mathrm{U}[0,2])$ holds.
- $\mathfrak{D}_2 = \mathfrak{C}(\mathrm{U}[0,2])$? We believe not.

Normalized admissible distributions

To let n vary, we normalize $\mathfrak{D}_n$ by $1/n$.

$$\mathfrak{G}_n(F) = \left\{ \text{cdf of } \frac{1}{n} S_n : X_i \sim F,\ i = 1, \ldots, n \right\}.$$

One can simply verify

Proposition 6

$\mathfrak{G}_n(F) \subset \mathfrak{G}_{kn}(F) \subset \mathfrak{C}(F)$ *for any* $k, n \in \mathbb{N}$ *and* $F \in \mathcal{F}^1$.

Normalized admissible distributions

For fixed $k \in \mathbb{N}$,

$$\mathfrak{G}_k(F) \subset \limsup_{n\to\infty} \mathfrak{G}_n(F) \subset \mathfrak{C}(F),$$

hence

$$\bigcup_{n\in\mathbb{N}} \mathfrak{G}_n(F) = \limsup_{n\to\infty} \mathfrak{G}_n(F) \subset \mathfrak{C}(F).$$

Theorem 7 (ongoing research with T. Mao)

For $F \in \mathcal{F}^1$,

$$\overline{\limsup_{n\to\infty} \mathfrak{G}_n(F)} = \mathfrak{C}(F).$$

where $\overline{A}$ the closure of $A \subset \mathcal{F}^1$ under the weak topology.

Sequence problem

In this part of the talk, assuming $X_i \sim F$, we consider a convergence problem of the following type:

$$\frac{S_n - a_n}{b_n} \to X \quad \text{in some sense of convergence as } n \to \infty,$$

for some real sequences $\{a_n,\ n \in \mathbb{N}\}$ and $\{b_n > 0,\ n \in \mathbb{N}\}$.

Classic settings

- Key elements:
  - (i) the marginal distribution F
  - (ii) the dependence structure of $\{X_i,\ i \in \mathbb{N}\}$
  - (iii) the limit X
    - The type of convergence and the normalizing constants are implied by the above elements.

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    - The type of convergence and the normalizing constants are implied by the above elements.
- Traditional study: given (i) and (ii), to find (iii), e.g. LLN, CLT, time series analysis.
- Another direction: given (ii) and (iii), to
  - determine whether they are compatible, e.g. stable distributions;
  - find (i), e.g. domains of attraction.

New question

My question: given (i) and (iii), to

- determine whether they are compatible, and
- find (ii).

Shapability

Denote by $\mathcal{F}$ the set of all distribution functions.

Definition 3

$F \in \mathcal{F}$ is said to be *shapable* to $G \in \mathcal{F}$ if there exist a sequence of random variables $\{X_i \sim F,\ i \in \mathbb{N}\}$, and real numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$, such that as $n \to \infty$,

$$\frac{S_n}{bn} - a \overset{\text{a.s.}}{\to} G.$$

We denote by $F \hookrightarrow G$.

Shapability

Consider the shapability as an order on equivalent classes in $\mathcal{F}$:

- Reflective: obviously $F \hookrightarrow F$.
- $\hookrightarrow$ transitive?
- $\hookrightarrow$ total?

Main result

Theorem 8 (W., 2014, preprint)

Suppose $F \in \mathcal{F}^1$ is non-degenerate. Then $F \hookrightarrow G$ for all bounded distributions G.

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Suppose $F \in \mathcal{F}^1$ is non-degenerate. Then $F \hookrightarrow G$ for all bounded distributions G.

Sketch of proof:

- Assume the mean of F and G are 0.
- Show $F \hookrightarrow G$ in the case of G is two-point using a conditional independence structure.
- Show that F is shapable to a mixture of G_a if $F \to G_a$ for each a, under some conditions.
- Write a general G as a mixture of two-point distributions with mean zero (not trivial).

Unbounded distributions

Denote $\mathcal{L}$ the set of regular varying distribution functions (not necessarily with the same index on both ends).

Theorem 9 (W., 2014, preprint)

Suppose $F, G \in \mathcal{F}^1$. *If* $F \hookrightarrow G$, *then*

$$\limsup_{t\to 1} \frac{G^{-1}(t)}{F^{-1}(t)} < \infty \text{ and } \limsup_{t\to 0} \frac{G^{-1}(t)}{F^{-1}(t)} < \infty. \quad (1)$$

If $F \in \mathcal{L}$ *or* $G \in \mathcal{L}$, *then* (1) *is sufficient for* $F \hookrightarrow G$.

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- $\hookrightarrow$ induces a total order on $\mathcal{L}$.

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If $F \in \mathcal{L}$ or $G \in \mathcal{L}$, then (1) *is sufficient for $F \hookrightarrow G$.*

- $\hookrightarrow$ induces a total order on $\mathcal{L}$.
- Small conjecture: (1) is sufficient and necessary in general.

Unbounded distributions

A slightly weaker condition than (1) is sufficient.

Proposition 10 (W., 2014, preprint)

Suppose $F, G \in \mathcal{F}^1$. *If*

$$\limsup_{t\to 1} \frac{G^{-1}(t)}{F^{-1}(t)} = 0 \quad \textit{and} \quad \limsup_{t\to 0} \frac{G^{-1}(t)}{F^{-1}(t)} = 0,$$

then $F \hookrightarrow G$.

Concluding message

The marginal constraint is very weak compared to the dependence uncertainty. If only marginal distributions are known, the sum can be of **any shape.**

- A support to the importance of studying copulas and dependence modeling.

Part IV - Challenges

Open mathematical questions:

- Does END sequence always exist in the homogeneous case for any marginal distributions?
- END/SEND construction for general unbounded $\{F_i,\ i \in \mathbb{N}\}$.
- Characterization of $\mathfrak{D}_n$.
- Necessary and sufficient conditions for $F \hookrightarrow G$ in general.
- Characterization of all dependence structures such that $F \hookrightarrow G$.
- Many more ...

Final remarks

I would be interested in the implication of the above results in

- Statistics?
- Quantitative risk management?
- Copulas?
- Extreme value theory?
- Monte-Carlo simulation?

Dependence matters a lot!

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Thank you

Thank you for your kind attendance

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