Risk Aggregation with Dependence Uncertainty

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Based on a series of joint work
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Risk and uncertainty:

- **Risk**: familiar; able to quantify; under control; quick response.

- **Uncertainty**: unfamiliar; difficult or impossible to quantify; beyond control; delayed response.
Part I - Introduction

Risk and uncertainty:

- **Risk**: familiar; able to quantify; under control; quick response.

- **Uncertainty**: unfamiliar; difficult or impossible to quantify; beyond control; delayed response.

**Model risk**: the risk of inappropriate modelling and misused quantitative tools.

- You think it is a risk but it is actually an uncertainty!
Risk aggregation

- $X_1, \cdots, X_n$ are random variables representing individual risks (one-period losses or profits).
- Aggregate position $S(X)$ associated with a risk vector $X = (X_1, \cdots, X_n)$.
- The most commonly used aggregation function is $S = X_1 + \cdots + X_n$. 
Challenges in dependence

- There is never perfect information. Statistical modelling and inference are needed.

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Marginal → risk; dependence → uncertainty.

The logic of using parameters, such as covariance matrices, Spearman's rho and tail dependence coefficients, to model dependence in risk management is questionable.
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Examples of model risk of dependence

Possibly misused modeling tools:

- Gaussian model.
- Conditional independence.
- Micro correlation.
- Independent increments.
- Behavior modeling.
April 23, 2013, S&P 500 index

What happened during those 10 minutes (1:07pm-1:16pm)?

Source: Yahoo finance
We seek a more general and mathematically tractable framework.
Part II - Dependence Uncertainty

We seek a more general and mathematically tractable framework.

- $S = X_1 + \cdots + X_n$.
- The marginal distributions of $X_1, \cdots, X_n$: known.
- The joint structure of $X_1, \cdots, X_n$: unknown.
- This setting is very practical.
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Target: probabilistic behavior of \( S \) and/or risk measures of \( S \).
Admissible risk class with uncertainty

For given univariate distributions $F_1, \cdots, F_n$, the admissible risk class (of marginals $F_1, \cdots, F_n$) is defined as

$$\mathcal{G}_n(F_1, \cdots, F_n) = \{ X_1 + \cdots + X_n : X_i \sim F_i, \ i = 1, \cdots, n \}. $$

Each $S \in \mathcal{G}_n(F_1, \cdots, F_n)$ is called an admissible risk (of marginals $F_1, \cdots, F_n$).
A few remarks

- \( \mathcal{G}_n(F) \) is the set of all possible aggregate risks when the marginal distributions are accurately obtained but the joint distribution is unknown.
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- $\mathcal{S}_n(F)$ is the set of all possible aggregate risks when the marginal distributions are accurately obtained but the joint distribution is unknown.

- The distribution of $S \in \mathcal{S}_n(F)$ is determined by the copula of $X_1, \cdots, X_n$. 
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- This admissible risk class has some nice theoretical properties, such as convexity w.r.t. distribution, permutation/affine/law-invariance, completeness, robustness.
A few remarks

- In practice, people may have partial information about the joint structure, such as
  - individual risks are positively quadratic dependent;
  - individual risks are conditional independent;
  - some information on the copula of $X$;
  - the covariance matrix is estimated accurately.

In those cases, the possible aggregate risks are in a subset of $\mathcal{G}_n(F)$. 
Remark on Fréchet classes

A Fréchet class:

\[ \mathcal{F}_n(F) := \{ (X_1, \cdots, X_n) : X_i \sim F_i, \ i = 1, \cdots, n \}. \]

The difference between \( S_n(F) \) and \( \mathcal{F}_n(F) \):

- The structure of \( \mathcal{F}_n(F) \) is marginal-independent, but \( S_n(F) \) is marginal-dependent.

- The information contained in \( \mathcal{F}_n(F) \) is redundant.
Questions on admissible risk classes

- Probabilistically, what exactly are in the set $\mathcal{G}_n(F)$?
  - For $S$ with a given distribution $F$, is $S$ in $\mathcal{G}_n(F)$? Is there a viable characterization?
  - What is the boundary (in some sense) of $\mathcal{G}_n(F)$?
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- Statistically, how can we conduct inference from data?
  - Traditional method: copula estimation - inaccurate, costly, provides information that are of no interest.
  - Direct estimation techniques: waste of marginal information.
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- Statistically, how can we conduct inference from data?
  - Traditional method: copula estimation - inaccurate, costly, provides information that are of no interest.
  - Direct estimation techniques: waste of marginal information.
- How can we use $\mathcal{S}_n(F)$ to manage risks?
  - Assign a measure on $\mathcal{S}_n(F)$? Risk $\Leftrightarrow$ uncertainty.
  - Extreme scenarios analysis?
  - Limited data regulation principles?
Part III - Extreme Scenarios

Extreme scenario questions for dependence uncertainty:

- Is a constant admissible?
- Convex ordering on admissible risks?
- Bounds for the distribution function of an admissible risk?
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Extreme scenario questions for dependence uncertainty:

- Is a constant admissible?
- Convex ordering on admissible risks?
- Bounds for the distribution function of an admissible risk?

These three questions turn out to be closely connected, via the concept of completely mixable distributions.
A few remarks

- Extreme scenarios $\rightarrow$ coherent measure of model uncertainty defined in Cont (2006):

$$
\mu_Q(\rho) = \sup_{Q \in \mathcal{Q}} \rho^Q(S) - \inf_{Q \in \mathcal{Q}} \rho^Q(S).
$$

- Research from the point of theoretical probability via a connection to mass-transportation can be found since early 80s, e.g. Rüschendorf (1982).

- A comprehensive overview on those topics can be found in the recent book Rüschendorf (2013).
Is a constant admissible?

- Basic observation: $\mathbb{E}[S]$ is a constant if $F_1, \cdots, F_n$ are $L_1$.
- Question: is a constant $K$, typically chosen as $\mathbb{E}[S]$, in $\mathcal{G}_n(F)$?
Joint Mixability

Joint mixable distributions (W., Peng and Yang, 2013)

We say the univariate distributions $F_1, \cdots, F_n$ are jointly mixable (JM) if there exists $X_i \sim F_i$, $i = 1, \cdots, n$ such that $X_1 + \cdots + X_n$ is a constant. Equivalently,

$$\mathcal{S}_n(F_1, \cdots, F_n) \cap \mathbb{R} \neq \emptyset.$$
Completely mixable distributions (Wang and W., 2011)

We say the univariate distribution $F$ is \textit{n-completely mixable} (CM) if there exists $X_1, \cdots, X_n \sim F$ such that $X_1 + \cdots + X_n$ is a constant. Equivalently,

$$\mathcal{E}_n(F, \cdots, F) \cap \mathbb{R} \neq \emptyset.$$
Interpretation of CM and JM:

- CM or JM scenarios represent a perfectly hedged portfolio.
- It is an ideal case of negative correlation. It is a natural generalization of the counter-comonotonicity ($n = 2$).
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- CM or JM scenarios represent a perfectly hedged portfolio.
- It is an ideal case of negative correlation. It is a natural generalization of the counter-comonotonicity \((n = 2)\).

An open research area:

**what distributions are CM/JM?**
Most relevant results for CM:

- If $F$ supported on $[a, b]$ with mean $\mu$ is $n$-CM, then the mean condition is necessary:

$$a + (b - a)/n \leq \mu \leq b - (b - a)/n.$$
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- The mean condition is sufficient for monotone densities.

- $\text{U}[0,1]$ is $n$-CM for $n \geq 2.$
Some fully characterized families:

- **Analytical proofs:**

- **Combinatorial proofs:**
Existing results for JM:

- **Generalized mean condition.**

- **Second order condition:** If $F_1, \cdots, F_n$ are JM with finite variance $\sigma_1^2, \cdots, \sigma_n^2$, then

  $$\max_{i \in \{1, \cdots, n\}} \sigma_i \leq \frac{1}{2} \sum_{i=1}^{n} \sigma_i.$$ 

- **W., Peng and Yang (2013):** the variance condition is sufficient for normal.

- **Wang and W. (2013a, preprint):** the variance condition is sufficient for uniform; elliptical; and unimodal-symmetric densities.
Mysteries of CM (JM)

- Uniqueness of the center?
- Unimodal densities and other types?
- Characterization?
- Asymptotic behavior \((n \to \infty)\)?
Convex ordering bounds

- We assume the individual risks are on $\mathbb{R}_+$ and are $L_1$ (finite mean).
- Since $\mathbb{E}[S]$ is fixed, the most interesting property is the convex order of $\mathcal{G}_n(F)$:

  For $X, Y \in L_1$, if $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for all convex functions $g : \mathbb{R} \rightarrow \mathbb{R}$, then we say $X \prec_{cx} Y$.

- In economics, the term second order stochastic dominance is more often used.
Why consider convex order?
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- Risk preference.
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- Coherent and convex risk measures.
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- $\mathbb{E}[g(S)]$:
  - expected utility;
  - the variance of aggregation, European basket option prices, realized variance options;
  - stop-loss premiums, losses with limits/deductibles.
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- Directly connects to bounds on the Value-at-Risk and optimal mass transportation problems.
- Mathematically nice and tractable.
Well-known results on this question (late 70s):

- The convex order upper bound is obtained by the **comonotonic scenario**: for \( S \in \mathcal{G}_n(F) \),

\[
S \prec_{cx} F_1^{-1}(U) + \cdots + F_n^{-1}(U)
\]

where \( U \) has a uniform distribution on \([0, 1] \).
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- The infimum:
  - Known for $n = 2$: **counter-monotonic scenario** for $S \in \mathcal{G}_2(F_1, F_2)$:

$$S \succeq_{cx} F_1^{-1}(U) + F_2^{-1}(1 - U).$$
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- All the above results are **marginal-independent**.
Connection between CM/JM distribution and convex ordering lower bound for $n \geq 3$:

- If $F_1, \cdots, F_n$ are JM, then $\mathbb{E}[S]$ is in $\mathcal{S}_n(F_1, \cdots, F_n)$, and thus it is the convex minimal element.

- CM/JM scenario is a natural generalization of the counter-cmonotonicity.

- Please note that the optimal structure is **marginal-dependent**. (I believe it is the reason why major progresses on this problem were delayed till recently.)
A surprising fact: for $n \geq 3$, the set $\mathcal{G}_n(F)$ may not contain a convex ordering minimal element.
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CM/JM is not possible for unbounded positive risks. We seek for more general results for the purpose of risk management:

- To obtain a convex minimal element, we try to enhance a density concentration (make $S$ as close to a constant as possible).

- Optimal structure for homogeneous marginals: tails - mutual exclusivity; body - complete mixability.
Analytical formulas for the lower bound on $\text{TVaR}_p(S)$ and $\mathbb{E}[g(S)]$ are available.

Lower bounds for heterogeneous marginals are obtained:

- not sharp in general, but quite accurate according to numerical results;
- the fact $\mathcal{G}_n(F_1, \cdots , F_n) \subset \mathcal{G}_n(F, \cdots , F)$ is used, where $F = \frac{1}{n} \sum_{i=1}^{n} F_i$. 

Bounds on the distribution function

Given marginal distributions, what is the maximum possible distribution function of $S$ (a special case of a question raised by A. N. Kolmogorov)?
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The question: given $F_1, \cdots, F_n$ and $s \in \mathbb{R}$, find

$$\sup_{S \in \mathcal{S}_n(F_1, \cdots, F_n)} \mathbb{P}(S \leq s) \quad \text{and} \quad \inf_{S \in \mathcal{S}_n(F_1, \cdots, F_n)} \mathbb{P}(S \leq s).$$
Equivalent question in risk management:

- Given $F_1, \cdots, F_n$ and $\alpha \in (0,1)$, find

\[
\sup_{S \in \mathcal{G}_n(F_1, \cdots, F_n)} \text{VaR}_\alpha(S) \quad \text{and} \quad \inf_{S \in \mathcal{G}_n(F_1, \cdots, F_n)} \text{VaR}_\alpha(S).
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- It is the best/worst scenario risk measure with confidence in marginal information.
- The usage of VaR in risk management is debatable for incoherence (non-subadditivity in particular) but still quite widely used.
- Very hard to solve analytically.
- What is done in the practice of operational risk: model marginal, add them up, and discount to 70%-90% due to *unjustified* diversification benefit.
Some literature

- Makarov (1981): \( n = 2 \).
- Rüschendorf (1982): independently solved \( n = 2 \).
- Identical marginals:
  - Rüschendorf (1982): dual representation; uniform and binomial cases.
  - W., Peng and Yang (2013): sharp bounds for homogeneous tail monotone densities based on CM.
  - Puccetti and Rüschendorf (2013): sharpness of dual bounds, equivalent to a CM condition.
Between VaR and convex ordering bounds

Suppose $F_1, \ldots, F_n$ are continuous distributions.

- $F_{i,a}$ for $a \in (0, 1)$ is the conditional distribution of $F_i$ on $[F_i^{-1}(a), \infty)$;
- $F_i^a$ for $a \in (0, 1)$ is the conditional distribution of $F_i$ on $(-\infty, F_i^{-1}(a))$. 
Theorem 1 (Bernard, Jiang and W. (2013))

(a) Suppose $S_a$ is a convex ordering minimum element in $\mathcal{G}_n(F_{1,a}, \cdots, F_{n,a})$ for $a \in (0, 1)$, then
$$\sup_{S \in \mathcal{G}_n(F_1, \cdots, F_n)} \text{VaR}_a(S) = \text{ess inf } S_a.$$ 

(b) Suppose $S_a$ is a convex ordering minimum element in $\mathcal{G}_n(F^a_1, \cdots, F^a_n)$ for $a \in (0, 1)$, then
$$\inf_{S \in \mathcal{G}_n(F_1, \cdots, F_n)} \text{VaR}_a(S) = \text{ess sup } S_a.$$
A few remarks:

- Finding convex ordering minimal element implies worst and best elements for VaR.
- The worst VaR only depends on the tail behavior, hence extra information on convariance/correlation may or may not affect its value.

Ruodu Wang
Part IV - Asymptotic Behavior

- Look at $S_n \in \mathcal{G}_n(F)$, $F = (F, \cdots, F)$, $F$ having mean $\mu$.
- When $F$ has finite second moment, we have looked at
  
  $$V_n = \text{Var}(S_n) \text{ and } \overline{V_n} = \inf_{S_n \in \mathcal{G}_n(F)} \text{Var}(S_n).$$

- What if $n \to \infty$?
Look at $S_n \in \mathcal{S}_n(F), F = (F, \cdots, F)$, $F$ having mean $\mu$.

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$$V_n = \text{Var}(S_n) \quad \text{and} \quad \underline{V}_n = \inf_{S_n \in \mathcal{S}_n(F)} \text{Var}(S_n).$$

What if $n \to \infty$?

- iid case: $V_n = O(n)$.
- comonotonic case: $V_n = O(n^2)$.
- what about most negative correlated case $\underline{V}_n$?
Variance reduction

**Theorem 2 (Wang and W. (2013b, preprint))**

*Suppose F has finite third moment then \( V_n = O(1) \).*
Variance reduction


Suppose $F$ has finite third moment then $V_n = O(1)$.

- A stronger result: there exists a sequence $X_i, \ i \in \mathbb{N}$ from $F$ such that $|S_n - n\mu| \leq Z$ a.s. for some $Z$ which does not depend on $n$. 

Suppose $F$ has finite third moment then $V_n = O(1)$.

- A stronger result: there exists a sequence $X_i, i \in \mathbb{N}$ from $F$ such that $|S_n - n\mu| \leq Z$ a.s. for some $Z$ which does not depend on $n$.
- For some $F$ this $O(1)$ is sharp, i.e. $V_n \not\to 0$. 
Asymptotic CM


*Suppose $F$ is supported in a finite interval with a strictly positive density function, then there exists $N \in \mathbb{N}$ such that $F$ is $n$-CM for all $n \geq N$.*

Asymptotically every distribution is (almost) CM.
Asymptotic equivalence

**Theorem 4**

*Under some conditions on $F$, for all $a \in (0, 1)$*

\[
\frac{\sup_{S \in \mathcal{G}_n(F, \ldots, F)} \text{VaR}_a(S)}{\sup_{S \in \mathcal{G}_n(F, \ldots, F)} \text{TVaR}_a(S)} \to 1.
\]

Worst VaR and worst TVaR (ES) are asymptotically equivalent.
• Puccetti and Rüschendorf (2013): $F$ is continuous, satisfies a conditional CM condition.

• Puccetti, Wang and W. (2013): $F$ is continuous and has strictly positive density based on CM.

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The same asymptotic equivalence holds for inhomogeneous marginals with very weak conditions on the marginal distributions.
Table: Values (rounded) for best- and worst VaR and ES for a homogeneous portfolio with $d$ Pareto(2) risks; $\alpha = 0.999$.

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<th>$d = 8$</th>
<th>$d = 56$</th>
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<tr>
<td><strong>Best VaR</strong></td>
<td>31</td>
<td>53</td>
</tr>
<tr>
<td><strong>Best TVaR</strong></td>
<td>145</td>
<td>472</td>
</tr>
<tr>
<td><strong>Coomonotonic VaR</strong></td>
<td>245</td>
<td>1715</td>
</tr>
<tr>
<td><strong>Worst VaR</strong></td>
<td>465</td>
<td>3454</td>
</tr>
<tr>
<td><strong>Worst TVaR</strong></td>
<td>498</td>
<td>3486</td>
</tr>
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In practice some people would use about $\text{VaR}_{\alpha}(S) \approx 200$ for $d = 8$ as the *conservative* capital reserve.
**Question.** Let $F, G$ be any two univariate distributions. Can you find random variables $X_i, i \in \mathbb{N}$ from $F$ such that $(S_n - a_n)/b_n \xrightarrow{d} G$ for some real sequences $a_n, b_n$?
**Shape problem**

**Question.** Let $F, G$ be any two univariate distributions. Can you find random variables $X_i, i \in \mathbb{N}$ from $F$ such that $(S_n - a_n)/b_n \overset{d}{\to} G$ for some real sequences $a_n, b_n$?

I think the answer is positive. The message is:

The marginal constraint is weak compared to the dependence uncertainty. If you only assume known marginals, you can end up with **anything**.
Part V - Challenges

- Theoretical results are basically unavailable for heterogeneous marginal distributions.
- Many unsolved mathematical problems.
- Applications in quantitative risk management.
Mathematical challenges

- Develop more classes of CM/JM distributions.
- Find sharp convex bounds for non-identical marginal distributions.
- Sufficient conditions for the existence of convex ordering minimal element in an admissible risk class?
- Improve numerical algorithms such as the Rearrangement Algorithm in Embrechts, Puccetti and Rüschendorf (2013).
Final remarks

- Practical risk management?
- Dynamic process?
- I expect connection with statistics and data science.
  - Modelling aggregate risks via estimating dependence structure may not be the best idea to study risk aggregation.
- Rather immature ideas; discussions are very much welcome.
References I


References II


References III


References IV


References V


Thank you for your attention
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or pretending

If you know what I mean