One axiom to rule them all: A minimalist axiomatization of quantiles

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Abstract

We offer a minimalist axiomatization of quantiles among all real-valued mappings on a general set of distributions through only one axiom. This axiom is called ordinality: quantiles are the only mappings that commute with all increasing and continuous transforms. Other convenient properties of quantiles, monotonicity, semicontinuity, comonotonic additivity, elicitability and locality in particular, follow from this axiom. Furthermore, on the set of convexly supported distributions, the median is the only mapping that commutates with all monotone and continuous transforms. On a general set of distributions, the median interval is pinned down as the unique minimal interval-valued mapping that commutes with all monotone and continuous transforms. Finally, our main result, put in a decision-theoretic setting, leads to a minimalist axiomatization of quantile preferences. In banking and insurance, quantiles are known as the standard regulatory risk measure Value-at-Risk (VaR), and thus, an axiomatization of VaR is obtained with only one axiom among law-based risk measures.

Key-words: quantiles; median; ordinality; quantile maximization; Value-at-Risk

1 Introduction

Quantiles are prominent objects in statistics, decision theory, optimization, mass transportation, machine learning, and finance. They have also been widely applied in the natural and social sciences as well as engineering through quantile regression and quantile optimization; see e.g., Koenker and Hallock (2001). In this paper, we offer a minimalist axiomatization of quantiles among all mappings on a general set of distributions via only one axiom.

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Let $\mathcal{X}$ be a set of random variables in a given atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, containing all bounded random variables. The left quantile at probability level $p \in (0, 1]$ is defined as

$$Q_p^L(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant p\}, \qquad X \in \mathcal{X},$$

where $\inf \emptyset = \infty$. The right quantile at probability level $p \in [0, 1)$ is defined as

$$Q_p^R(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) > p\}, \qquad X \in \mathcal{X}.$$

We consider law-based mappings $\mathcal{R}$ on $\mathcal{X}$, meaning that $\mathcal{R}(X)$ is determined by the distribution of X.[1] Throughout, terms like "increasing" are in the non-strict sense.

Theorem 1. *For a law-based mapping $\mathcal{R} : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) $\mathcal{R} \circ \phi = \phi \circ \mathcal{R}$ for all increasing and continuous functions $\phi : \mathbb{R} \to \mathbb{R}$;

(ii) $\mathcal{R}$ is a quantile; that is, $\mathcal{R} = Q_p^L$ for some $p \in (0, 1]$ or $\mathcal{R} = Q_p^R$ for some $p \in [0, 1)$.

The commutation property in Theorem 1 (i) will be called $\mathcal{G}^*$-*ordinality* (formal definition in Section 2), and $\mathcal{R} \circ \phi = \phi \circ \mathcal{R}$ means that $\mathcal{R}(\phi(X)) = \phi(\mathcal{R}(X))$ for all $X \in \mathcal{X}$ such that $\phi(X)$ remains in $\mathcal{X}$. An alternative version of Theorem 1 on a domain of distributions is formulated as Theorem 2 in Section 3. We will further show that if $\mathcal{G}^*$-ordinality is slightly strengthened, then left and right quantiles can be distinguished, i.e., commutating with all left-continuous and increasing functions leads to a left quantile, and commutating with all right-continuous and increasing functions yields a right quantile (Proposition 1).

The interpretation of $\mathcal{G}^*$-ordinality in decision making is intuitive: For an index which quantifies random objects, a possibly non-linear scale change in the random outcomes gives rise to the same scale change on the index; see e.g., Chambers (2007) in the context of utility aggregation. We make this interpretation rigorous in a decision-theoretic setting in Section 5.

Theorem 1 is an extension of the main result of Chambers (2009), who showed that the left quantile is the only function that satisfies the three axioms of monotonicity (with respect to first-order stochastic dominance), lower semicontinuity, and ordinal covariance (commuting with strictly increasing and continuous functions). Contrasting Theorem 1 with that result, our $\mathcal{G}^*$-ordinality is slightly stronger than ordinal covariance imposed by Chambers (2009), and monotonicity and lower semicontinuity are not needed. Moreover, results of Chambers (2009) are

[1] Law-based mappings with a specified probability $\mathbb{P}$ are common in the axiomatic theory of preferences, risk measures, and statistical functionals; see e.g., the classic studies von Neumann and Morgenstern (1947), Quiggin (1982), Yaari (1987) and Kusuoka (2001) or the more recent Mu et al. (2021). The law-based property is also called probabilistic sophistication (Machina and Schmeidler (1992)) with respect to the probability $\mathbb{P}$, either objective or subjective. In the risk measure literature, the property is often called law invariance (Föllmer and Schied (2016)).

obtained on either $\mathcal{X} = L^\infty$ (the set of essentially bounded random variables) or $\mathcal{X} = L^0$ (the set of all random variables) whereas our result holds on any domain $\mathcal{X}$ with $L^\infty \subseteq \mathcal{X} \subseteq L^0$. Despite the obvious similarity, our extension is by no means technically straightforward. Enlarging the set of functions that commute with $\mathcal{R}$ makes the corresponding ordinality property stronger, but choosing such an enlargement requires subtle sophistication. For instance, one may be tempted to require $\mathcal{R}$ to commute with all increasing functions, but such $\mathcal{R}$ does not exist; see Example 1 below. Therefore, extra care has to be taken when formulating the set which $\mathcal{R}$ commutes with, and this may partially explain why the minimalist characterization in Theorem 1 was not found in the literature. Moreover, our proof techniques are completely different from Chambers (2009).

Example 1. There is no such $\mathcal{R} : L^\infty \to \mathbb{R}$ satisfying $\mathcal{R} \circ \phi = \phi \circ \mathcal{R}$ for all increasing function ϕ. To see this, suppose otherwise. Take any continuously distributed random variable $X \in L^\infty$. Let $\phi_1(x) = \mathbb{1}_{\{x \geqslant \mathcal{R}(X)\}}$ and $\phi_2(x) = \mathbb{1}_{\{x > \mathcal{R}(X)\}}$, where $\mathbb{1}$ is the indicator function. Note that $\phi_1(X) = \phi_2(X)$ almost surely (and thus they are equal in L^∞). A contradiction arises as $\mathcal{R}(\phi_1(X)) = \phi_1(\mathcal{R}(X)) = 1 > 0 = \phi_2(\mathcal{R}(X)) = \mathcal{R}(\phi_2(X)) = \mathcal{R}(\phi_1(X))$.

If one concentrates on strictly increasing distribution functions (equivalently, distributions with a convex support), then $\mathcal{G}^*$-ordinality, which becomes weaker in this setting due to the smaller domain, is able to characterize quantiles on such a domain (Proposition 2); if one considers strictly increasing and continuous distribution functions, then a weaker ordinality property is sufficient. Commutation with continuous monotone (i.e., increasing or decreasing) functions (this is called $\mathcal{G}^*_\pm$-ordinality) further pins down the median on sets of convexly supported distributions (Theorem 3). As the median is not unique for general distributions, we consider median intervals, which are characterized as the minimal interval-valued mappings satisfying $\mathcal{G}^*_\pm$-ordinality (Theorem 4).

Since quantiles are characterized by $\mathcal{G}^*$-ordinality alone, all the other nice properties of quantiles, such as monotonicity, constant additivity, positive homogeneity, continuity, comonotonic additivity (Yaari (1987); Schmeidler (1989)), elicitability (Lambert et al. (2008); Gneiting (2011); Ziegel (2016)),[2] tail relevance (Liu and Wang (2021)), and locality (Bellini and Peri (2022); see Remark 1) are obtained for free. In particular, it may be surprising that monotonicity is implied, noting that it does not follow from ordinal covariance used by Chambers (2009).[3]

[2]Rigourously speaking, the quantile interval $[Q_p^L, Q_p^R]$ is elicitable; see Section 4.

[3]An observation in a similar fashion is made by Wang and Wei (2020), in which a comonotonic-additive, elicitable and uniformly continuous mapping on L^∞ is necessarily monotone (either increasing or decreasing), although no single one of these properties implies monotonicity by itself. Quantiles are a special class of such mappings.

Why is monotonicity not needed in our characterization? For a continuous and increasing function f satisfying $f(x) \geqslant x$ for $x \in \mathbb{R}$, we have $f(X) \geqslant X$ for $X \in \mathcal{X}$, and $\mathcal{G}^*$-ordinality gives $\mathcal{R}(f(X)) = f(\mathcal{R}(X)) \geqslant \mathcal{R}(X)$. Therefore, some weak form of monotonicity, which only applies to random pairs of the form $(X, f(X))$, is guaranteed by $\mathcal{G}^*$-ordinality. A technical obstacle is to extend this observation to general random variables, which turns out to be a relatively challenging task, overcome by the proof of Theorem 1.

The minimalist axiomatic foundation for quantiles are useful in many other contexts, as quantiles are a useful and important alternative to the standard utility theory in decision making. Quantiles serve as the building blocks for the dual utility of Yaari (1987) since a dual utility can be written as a mixture of quantiles. Preferences induced by quantiles are axiomatized by Rostek (2010) and later by de Castro and Galvao (2022) in Savage and Anscombe-Aumann settings, both without a pre-specified probability space. Applying Theorem 1, we obtain a concise axiomatic foundation for quantile maximization among law-based preferences, characterized by $\mathcal{G}^*$-ordinality and certainty equivalents (Theorem 5).

There are many other important applications of quantiles in economics and finance. For instance, Basak and Shapiro (2001) studied utility maximization with quantile constraints, Embrechts et al. (2018) studied collaborative and competitive equilibria for quantile agents in risk sharing games, and de Castro and Galvao (2019) developed a dynamic model for a rational quantile maximizer. Empirical evidence for decision making with quantile optimization is studied by de Castro et al. (2022). In financial risk management, quantiles are known as Value-at-Risk (VaR), the dominating risk measure in banking and insurance over the past two decades. Although Expected Shortfall (ES) was proposed to replace VaR in the recent Basel Accords, VaR is still widely applied in regulatory capital calculation, decision making, performance analysis, and backtesting. We refer to McNeil et al. (2015) for a general treatment on VaR and other risk measures. Because of the practical importance of VaR, there are a few other sets of axioms for VaR, or quantiles, in addition to Chambers (2009). With some other standard properties including monotonicity, VaR is characterized via elicitability and comonotonic additivity by Kou and Peng (2016), via surplus invariance by He and Peng (2018), and via elicitability and tail relevance by Liu and Wang (2021). On the other hand, ES is axiomatized by Wang and Zitikis (2021) with a property reflecting portfolio diversification. In each result above, a class of risk measures is characterized by at least three properties, whereas Theorem 1 axiomatizes quantiles using only one. As far as we are aware, our result is the first characterization of quantiles that does not assume monotonicity.

All proofs of results are collected in Appendix A. For the rest of the paper, we conveniently formulate all mappings with domains being sets of distributions instead of random variables.

2 Model setting

Let $\mathcal{M}_0$ be the set all distributions on $\mathbb{R}$ and $\mathcal{M}_c$ be the set of all compactly supported distributions in $\mathcal{M}_0$. Throughout the paper, $\mathcal{M}$ is the domain of a mapping of interest, satisfying $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$. Distributions in $\mathcal{M}$ will be identified with their cdfs. For a cdf $F \in \mathcal{M}$, we define its left quantile function as

$$F_L^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geqslant t\}, \quad t \in (0,1], \tag{1}$$

and its right quantile function as

$$F_R^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) > t\}, \quad t \in [0,1). \tag{2}$$

An important object in the paper is the shape transform. For a measurable function $\phi : \mathbb{R} \to \mathbb{R}$, the *shape transform* $T^{[\phi]} : \mathcal{M} \to \mathcal{M}$ is defined as a mapping from the distribution of a random variable X to the distribution of $\phi(X)$, that is, $T^{[\phi]}(F) = F \circ \phi^{-1}$, where F is treated as a measure on $\mathbb{R}$ and $\phi^{-1}(A) = \{x \in \mathbb{R} : \phi(x) \in A\}$ for any Borel set $A \subseteq \mathbb{R}$. The transform function ϕ will always be monotone in this paper, which means that it is either increasing or decreasing. In the context of distributional transforms, which are mappings from $\mathcal{M}$ to $\mathcal{M}$, Liu et al. (2021) showed that shape transforms can be used to characterize probability distortions.

In the remainder of the paper, different from Section 1, we will conveniently treat law-based mappings as functions from $\mathcal{M}$ to $\mathbb{R}$. We first define ordinality, the most important property in this paper.

Definition 1. For a mapping $\rho : \mathcal{M} \to \mathbb{R}$ and a set $\mathcal{G}$ of measurable functions, we say that ρ is $\mathcal{G}$-*ordinal* if $\rho \circ T^{[\phi]} = \phi \circ \rho$ for all $\phi \in \mathcal{G}$. Here, $\rho \circ T^{[\phi]} = \phi \circ \rho$ means that $\rho(T^{[\phi]}(F)) = \phi(\rho(F))$ for all $F \in \mathcal{M}$ such that $T^{[\phi]}(F) \in \mathcal{M}$.

We say $\rho : \mathcal{M} \to \mathbb{R}$ is a *left quantile* if there exists $p \in (0,1]$ such that $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}$; $\rho : \mathcal{M} \to \mathbb{R}$ is a *right quantile* if there exists $p \in [0,1)$ such that $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}$. Note that ρ is assumed to be real-valued on $\mathcal{M}$. The value $p = 1$ needs to be excluded for left quantiles if $\mathcal{M}$ contains F such that $F_L^{-1}(1) = \infty$. The value $p = 0$ needs to be excluded for right quantiles if $\mathcal{M}$ contains F such that $F_R^{-1}(0) = -\infty$. For instance, both would be excluded if $\mathcal{M} = \mathcal{M}_0$.

Different choices of $\mathcal{G}$ in $\mathcal{G}$-ordinality lead to different results on different domains $\mathcal{M}$. We collect in Table 1 a summary of definitions of different sets of distributions $\mathcal{M}$ and sets of functions $\mathcal{G}$, which will be used throughout the paper. The most important choice of $\mathcal{G}$ is $\mathcal{G}^*$, the set of continuous and increasing real functions. The classes $\mathcal{M}$ of distributions can be described using either properties on the distribution function or, equivalently, those on the quantile function, and we state such equivalence in Table 1.

set	left or right quantile functions on $(0,1)$	distribution functions on its support
$\mathcal{M}_0$	all	all
$\mathcal{M}_c$	bounded	compactly supported (c.s.)
$\mathcal{M}_0^*$	continuous	strictly increasing
$\mathcal{M}_c^*$	bounded and continuous	c.s. and strictly increasing
$\mathcal{M}_0^\diamond$	continuous and strictly increasing	continuous and strictly increasing
$\mathcal{M}_c^\diamond$	bounded, continuous and strictly increasing	c.s., continuous and strictly increasing

set	transform functions ϕ on $\mathbb{R}$
$\mathcal{G}^*$	ϕ is increasing and continuous
$\mathcal{G}_\pm^*$	ϕ is monotone and continuous
$\mathcal{G}^L$	ϕ is increasing and left-continuous
$\mathcal{G}^R$	ϕ is increasing and right-continuous
$\mathcal{G}^\diamond$	ϕ is strictly increasing and continuous
$\mathcal{G}_\pm^\diamond$	ϕ is strictly monotone and continuous

Table 1: A summary of notation

3 Ordinality axiomatizes quantiles

We first present our main result on the characterization of quantiles based on $\mathcal{G}^*$. Theorem 1 in the Introduction follows directly from Theorem 2 by translating results on $\mathcal{M}$ to those on $\mathcal{X}$.

Theorem 2. *A mapping $\rho : \mathcal{M} \to \mathbb{R}$ with $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$ is $\mathcal{G}^*$-ordinal if and only if it is a left or right quantile.*

We explain here a sketch of the proof of Theorem 2 in the case of a compact support ($\mathcal{M} = \mathcal{M}_c$) for the interested reader, and a detailed proof is put in Appendix A. The "if" part can be checked by the definition of quantiles, although it requires some derivation. For the "only if" part, we will first show that for every $F \in \mathcal{M}_c$, $\rho(F)$ belongs to the range of F_L^{-1} or F_R^{-1}. As a consequence, $\rho(F)$ is equal to the left or right quantile at a probability level p_F depending on F. Note that the quantile type (left or right quantile) also depends on F. It remains to show that the probability p_F and the quantile type are the same for each $F \in \mathcal{M}_c$. Note that $\mathcal{M}_c^*$ (i.e., the set of compactly supported and strictly increasing distributions) can be generated from a standard uniform distribution F_U through shape transforms in $\mathcal{G}^*$, i.e., $\mathcal{M}_c^* = \{T^{[\phi]}(F_U) : \phi \in \mathcal{G}^*\}$. Using $\mathcal{G}^*$-ordinality, we can now conclude that the probability p_F

and the quantile type are the same for each $F \in \mathcal{M}_c^*$. The remaining challenge is to extend this observation from $\mathcal{M}_c^*$ to $\mathcal{M}_c$, noting that $\mathcal{M}_c$ cannot be generated by shape transforms from a distribution $F \in \mathcal{M}_c$; that is, we cannot find F so that $\mathcal{M}_c = \{T^{[\phi]}(F) : \phi \in \mathcal{G}^*\}$. We show this extension by two steps. We first show that the result also holds on $\mathcal{M}^I$, the set of all continuous distributions, as $T^{[F]}(F) = F_U$ for all $F \in \mathcal{M}^I$. Next, for $F \in \mathcal{M}_c \setminus (\mathcal{M}_c^* \cup \mathcal{M}^I)$, we show by a delicate analysis that F can be connected to distributions in $\mathcal{M}_c^* \cup \mathcal{M}^I$ in one of the following two ways: i) there exists $\phi \in \mathcal{G}^*$ such that $T^{[\phi]}(F) \in \mathcal{M}_c^* \cup \mathcal{M}^I$; ii) there exists $G \in \mathcal{M}_c^* \cup \mathcal{M}^I$ and $\phi \in \mathcal{G}^*$ such that $T^{[\phi]}(F) = T^{[\phi]}(G)$. These connections and $\mathcal{G}^*$-ordinality ensure the uniqueness of p_F and the quantile type on $\mathcal{M}_c$.

To characterize quantile using ordinality as in Theorem 2, the set $\mathcal{G}^*$ cannot be replaced by the smaller one $\mathcal{G}^\diamond$ used by Chambers (2009), as shown by the following example.

Example 2. For two distinct $p, q \in (0,1)$, define

$$\rho(F) = \begin{cases} F_L^{-1}(p), & F \in \mathcal{M}_c^* \\ F_L^{-1}(q), & F \in \mathcal{M}_c \setminus \mathcal{M}_c^*. \end{cases}$$

For $\phi \in \mathcal{G}^\diamond$, $\mathcal{M}_c^*$ and $\mathcal{M}_c \setminus \mathcal{M}_c^*$ are both closed under $T^{[\phi]}$. Hence, ρ satisfies $\rho \circ T^{[\phi]} = \phi \circ \rho$ for $\phi \in \mathcal{G}^\diamond$ but ρ is not a quantile. This example also shows that $\mathcal{G}^\diamond$-ordinality does not imply monotonicity.

In Theorem 2, if the set of transforms $\mathcal{G}^*$ is enlarged to $\mathcal{G}^L$ or $\mathcal{G}^R$ (i.e., including either the left-continuous or the right-continuous ones), then we obtain the characterization of left and right quantiles, respectively.

Proposition 1. *Let $\rho : \mathcal{M} \to \mathbb{R}$ where $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$. We have*

(i) ρ is $\mathcal{G}^L$-ordinal if and only if ρ is a left quantile;

(ii) ρ is $\mathcal{G}^R$-ordinal if and only if ρ is a right quantile.

On a set $\mathcal{M}$ of distributions with a convex support (these distributions have strictly increasing distribution functions on their support, i.e., $\mathcal{M}$ is a subset of $\mathcal{M}_0^*$), each distribution has a continuous quantile function, and thus its left and right quantiles coincide on $(0,1)$, and we do not distinguish them. Further, we can consider the domain of distributions with continuous and strictly increasing quantiles, i.e., subsets of $\mathcal{M}_0^\diamond$. Clearly, $\mathcal{M}_c^\diamond \subsetneq \mathcal{M}_c^* \subsetneq \mathcal{M}_c$ and $\mathcal{M}_0^\diamond \subsetneq \mathcal{M}_0^* \subsetneq \mathcal{M}_0$. Note that the property of having a continuous and strictly increasing quantile is satisfied by any distribution with a positive density function on its support.

As such a choice of $\mathcal{M}$ is not between $\mathcal{M}_c$ and $\mathcal{M}_0$, the characterization of quantiles on $\mathcal{M}$ does not directly follow from Theorem 2, which requires $\mathcal{M} \supseteq \mathcal{M}_c$ as a condition. Nevertheless, the corresponding arguments for $\mathcal{G}^*$-ordinality on $\mathcal{M}_c^*$ can be found in the proof of Theorem 2. For the smaller sets between $\mathcal{M}_c^\diamond$ and $\mathcal{M}_0^\diamond$ of more regularized distributions, it turns out that $\mathcal{G}^\diamond$-ordinality is sufficient, that is, the set of strictly increasing and continuous transforms, which is a subset of $\mathcal{G}^*$.

Proposition 2. *For a mapping $\rho : \mathcal{M} \to \mathbb{R}$,*

(i) if $\mathcal{M}_c^* \subseteq \mathcal{M} \subseteq \mathcal{M}_0^*$, then ρ is $\mathcal{G}^*$-ordinal if and only if it is a quantile;

(ii) if $\mathcal{M}_c^\diamond \subseteq \mathcal{M} \subseteq \mathcal{M}_0^\diamond$, then ρ is $\mathcal{G}^\diamond$-ordinal if and only if it is a quantile.

Another observation is that adding the decreasing and continuous functions to $\mathcal{G}^*$ will lead to the median. The median is of central importance in robust statistics, and it is a popular quantitative tool in many domains of application, such as assessing income inequality, due to its many advantages over other central statistics such as the mean or the mode. As far as we are aware, an axiomatic characterization for the median is still missing in the literature, and the next theorem fills in this gap. Note that the median is uniquely defined on $\mathcal{M}_0^*$ but not $\mathcal{M}_0$, and for this reason we will work with subsets of $\mathcal{M}_0^*$.

Theorem 3. *For a mapping $\rho : \mathcal{M} \to \mathbb{R}$,*

(i) if $\mathcal{M}_c^* \subseteq \mathcal{M} \subseteq \mathcal{M}_0^*$, then ρ is $\mathcal{G}_\pm^*$-ordinal if and only if it is the median;

(ii) if $\mathcal{M}_c^\diamond \subseteq \mathcal{M} \subseteq \mathcal{M}_0^\diamond$, then ρ is $\mathcal{G}_\pm^\diamond$-ordinal if and only if it is the median.

The characterization of the median in Theorem 3 follows from the fact the median is the only quantile on $\mathcal{M}_c^*$ satisfying *antisymmetry*, i.e., commutation with the negative identity function. However, Theorem 3 does not hold on a set of distributions with discontinuous quantiles, such as $\mathcal{M}_c$. This is because none of the left-median and the right-median on $\mathcal{M}_c$ satisfies antisymmetry. Note that antisymmetry and $\mathcal{G}^*$-ordinality together are equivalent to $\mathcal{G}_\pm^*$-ordinality. In contrast, the median is unique on $\mathcal{M}_c^*$, and antisymmetry holds.

Before ending this section, we offer another characterization of quantiles, with the help of a continuity condition. Below, $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$.

Proposition 3. *For a mapping $\rho : \mathcal{M} \to \mathbb{R}$ with $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$, ρ is a left or right quantile if and only if it satisfies*

(i) $\rho \circ T^{[\phi]} = \phi \circ \rho$ for all functions $\phi : x \mapsto x + c$ where $c \in \mathbb{R}$;

(ii) $\rho \circ T^{[\phi]} = \phi \circ \rho$ for all functions $\phi : x \mapsto ax_+ - bx_-$ where $a, b \geqslant 0$;

(iii) ρ is continuous with respect to the sup-norm on quantile functions.

The main intuition behind Proposition 3 is that any continuous and increasing function on a bounded interval can be uniformly approximated by piece-wise linear functions, which can in turn be obtained by applying the transforms in (i) and (ii) repeatedly. Property (i) in Proposition 3 is known as cash additivity, which is a common property of monetary risk measures (e.g., Föllmer and Schied (2016)). Property (ii) requires that ρ commutes with piece-wise linear transforms that apply different scaling to the positive and the negative parts of the random variable; this property is quite strong, and satisfied by e.g., $\rho : F \mapsto \int x_+ \mathrm{d}F$ and $\rho : F \mapsto \int x_- \mathrm{d}F$. Property (iii) is a standard continuity which is satisfied by all law-based monetary risk measures.[4] Continuity of ρ is not assumed in any other results in this paper.

Remark 1. Bellini and Peri (2022) proposed the locality property and used it to characterize a class of risk measures called the Λ-VaR. A mapping $\rho : \mathcal{M} \to \mathbb{R}$ is *local* if for any interval $(x, y) \subseteq \mathbb{R}$, $\rho(F) \in (x, y)$ implies $\rho(F) = \rho(G)$ for all $G \in \mathcal{M}$ with $G = F$ on (x, y). Locality can be directly derived from $\mathcal{G}^*$-ordinality via the following argument. Note that the function $\phi : x \mapsto a \vee x \wedge b$ for $a < b$ is in $\mathcal{G}^*$. Suppose that $\rho(F) \in (a, b)$ and $F = G$ on (a, b). We have $T^{[\phi]}(F) = T^{[\phi]}(G)$ since $F \circ \phi^{-1} = G \circ \phi^{-1}$. Now, $\mathcal{G}^*$-ordinality gives $\phi(\rho(F)) = \phi(\rho(G))$. Hence, locality follows from $\mathcal{G}^*$-ordinality.

4 Quantile and median intervals

We next consider the characterization of quantile intervals on $\mathcal{M}$. Instead of a real number, the median for a distribution without a continuous quantile may be an interval. We denote by $\mathbb{I} = \{[a, b] : -\infty < a \leqslant b < \infty\}$, that is, the set of all compact intervals on $\mathbb{R}$. We consider mappings $\rho : \mathcal{M} \to \mathbb{I}$. The property of ordinality is adapted as follows.

Definition 2. For a mapping $\rho : \mathcal{M} \to \mathbb{I}$ and a set $\mathcal{G}$ of measurable functions, we say that ρ is *$\mathcal{G}$-ordinal* if $\rho \circ T^{[\phi]} = \phi \circ \rho$ for all $\phi \in \mathcal{G}$. Here, $\rho \circ T^{[\phi]} = \phi \circ \rho$ means that $\rho(T^{[\phi]}(F)) = \phi(\rho(F))$ for all $F \in \mathcal{M}$ such that $T^{[\phi]}(F) \in \mathcal{M}$, where $\phi(I) = \{\phi(x) : x \in I\}$ for $I \in \mathbb{I}$.

We say that $\rho : \mathcal{M} \to \mathbb{I}$ is a *quantile interval* if both its endpoints are quantiles. More precisely, there exist $0 \leqslant p_1 \leqslant p_2 \leqslant 1$ such that one of the following four scenarios holds: (a) $\rho(F) = [F_L^{-1}(p_1), F_R^{-1}(p_2)]$ for all $F \in \mathcal{M}$; (b) $\rho(F) = [F_R^{-1}(p_1), F_L^{-1}(p_2)]$ for all $F \in \mathcal{M}$; (c) $\rho(F) = [F_L^{-1}(p_1), F_L^{-1}(p_2)]$ for all $F \in \mathcal{M}$; (d) $\rho(F) = [F_R^{-1}(p_1), F_R^{-1}(p_2)]$ for all $F \in \mathcal{M}$.[5]

[4] To see this claim, let $F, G \in \mathcal{M}$ and $c = \sup_{t\in(0,1)} |F_L^{-1}(t) - G_L^{-1}(t)|$, yielding $G_L^{-1} - c \leqslant F_L^{-1} \leqslant G_L^{-1} + c$. As a monetary risk measure is monotone and cash additive, we have $|\rho(F) - \rho(G)| \leqslant c$, which implies property (iii).

[5] Note that for case (b), $p_1 < p_2$ is required and for cases (a)-(d), $p_2 = 1$ in excluded if there exists $F \in \mathcal{M}$ such that $F_L^{-1}(1) = \infty$, and $p_1 = 0$ is excluded if there exists $F \in \mathcal{M}$ such that $F_R^{-1}(0) = -\infty$.

A quantile interval ρ is *equal-tailed* if $\rho(F) = [F_L^{-1}(p), F_R^{-1}(1-p)]$ for $p \in (0, 1/2]$ or $\rho(F) = [F_R^{-1}(p), F_L^{-1}(1-p)]$ for $p \in (0, 1/2)$.

Proposition 4. *For a mapping $\rho : \mathcal{M} \to \mathbb{I}$ with $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$,*

(i) ρ is $\mathcal{G}^*$-ordinal if and only if it is a quantile interval;

(ii) ρ is $\mathcal{G}^*_{\pm}$-ordinal if and only if it is an equal-tailed quantile interval.

For two interval functions $\rho : \mathcal{M} \to \mathbb{I}$ and $\rho' : \mathcal{M} \to \mathbb{I}$, we say that ρ contains ρ' if $\rho'(F) \subseteq \rho(F)$ for all $F \in \mathcal{M}$. For a set $\mathcal{G}$ of measurable functions, we say that $\rho : \mathcal{M} \to \mathbb{I}$ is *minimally $\mathcal{G}$-ordinal* if ρ is $\mathcal{G}$-ordinal and it contains no other $\mathcal{G}$-ordinal function $\rho' : \mathcal{M} \to \mathbb{I}$. Intuitively, minimal $\mathcal{G}$-ordinality leads to smallest intervals, possibly a singleton. Combination of Proposition 4 and the above definition, we immediately arrive at the following characterization results.

Theorem 4. *For a mapping $\rho : \mathcal{M} \to \mathbb{I}$ where $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$,*

(i) ρ is minimally $\mathcal{G}^*$-ordinal if and only if it is the singleton of a left or right quantile;

(ii) ρ is minimally $\mathcal{G}^*_{\pm}$-ordinal if and only if it is the median interval.

Note that minimal $\mathcal{G}^*_{\pm}$-ordinality does not imply, and is not implied by, minimal $\mathcal{G}^*$-ordinality, and the corresponding characterized mappings are non-overlapping. Theorem 4 uniquely pins down the median interval among all interval-valued mappings on $\mathcal{M}$ as the smallest interval which satisfies antisymmetry and $\mathcal{G}^*$-ordinality. Note that in Theorem 4, we see that, without antisymmetry, the minimal $\mathcal{G}^*$-ordinality does not pin down quantile intervals of the form $[F_L^{-1}(p), F_R^{-1}(p)]$. Hence, median interval indeed has a unique role among all quantile intervals, not only because of its antisymmetry, but also because of its minimality.

The mapping $F \mapsto [F_L^{-1}(p), F_R^{-1}(p)]$ for $p \in (0,1)$, which we call the *p-quantile interval*, is also special among quantile intervals. It appears that such intervals cannot be characterized with only ordinal properties. For a characterization of this class, we need an additional property. A mapping $\rho : \mathcal{M} \to \mathbb{I}$ is *elicitable* if there exists a function $S : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho(F) = \arg\min_{x \in \mathbb{R}} \int_{\mathbb{R}} S(x,y) \mathrm{d}F(y)$ for all $F \in \mathcal{M}$. Such function S is called a score function for ρ. It is well known that the p-quantile interval is elicitable with the score function $S(x,y) = (\mathbb{1}_{\{x \geqslant y\}} - p)(x-y)$; see Gneiting (2011) for this and other results on elicitability.

Proposition 5. *For a mapping $\rho : \mathcal{M} \to \mathbb{I}$ with $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$,*

(i) ρ is $\mathcal{G}^*$-ordinal and elicitable if and only if it is the p-quantile interval for some $p \in (0,1)$;

(ii) ρ is $\mathcal{G}^*_{\pm}$-ordinal and elicitable if and only if it is the median interval.

To prove Proposition 5, we will check that the other types of quantile intervals in Proposition 4 do not satisfy elicitability. Our definition of elicitability, following Gneiting (2011), is different from that of Brehmer and Gneiting (2021) for interval mappings, where each interval is treated as a two-dimensional vector.

5 Axiomatization of quantile maximization

Quantile maximization was axiomatized by Rostek (2010) in decision theory with several Savage-style axioms. An alternative axiomatization is obtained by de Castro and Galvao (2022) using ordinal covariance on a finite state space. Both above theories are different from our framework as we consider mappings and preferences in a pre-specified probability space.[6] Our results in Section 3 lead to an axiomatic characterization for quantile maximization on general sets with a minimal number of axioms.

Let $\mathcal{X}$ be a set of random variables containing all bounded random variables. Each element of $\mathcal{X}$ represents a random prospect, and we treat almost surely equal random variables as identical. As discussed by Chambers (2007), the interpretation of $\mathcal{G}$-ordinality in decision making is that a possibly non-linear scale change in $\mathcal{G}$ on measuring random outcomes does not affect their relative desirability. Putting this into a decision-theoretic framework, let the preference $\preceq$ be a total preorder on $\mathcal{X}$[7] that is law-based. Its equivalence relation and strict relation are denoted by $\sim$ and $\prec$, respectively. The natural formulation of $\mathcal{G}$-ordinality in this context, which will be called *$\mathcal{G}$-invariance* of $\preceq$, is

$$X \preceq Y \implies \phi(X) \preceq \phi(Y) \text{ for all } \phi \in \mathcal{G}. \tag{3}$$

If $\preceq$ is represented by a functional $\mathcal{R} : \mathcal{X} \to \mathbb{R}$,[8] then the property becomes $\mathcal{G}$-invariance of $\mathcal{R}$, that is,

$$\mathcal{R}(X) \leqslant \mathcal{R}(Y) \implies \mathcal{R}(\phi(X)) \leqslant \mathcal{R}(\phi(Y)) \text{ for all } \phi \in \mathcal{G}. \tag{4}$$

The $\mathcal{G}$-invariance property in (4) is genuinely weaker than $\mathcal{G}$-ordinality. For instance, $\mathcal{G}^*$-invariance is satisfied by the constant map $\mathcal{R} = 0$ or any monotone transform of a quantile functional, and they do not satisfy $\mathcal{G}^*$-ordinality in general.

We say that a preference relation $\preceq$ *has certainty equivalents* if for every $X \in \mathcal{X}$ there exists a constant $c \in \mathbb{R}$ such that $X \sim c$. We say that a preference relation $\preceq$ is *quantile maximizing*

[6] Such a setting is common in the literature of statistics and risk management, in particular on risk measures or statistical functionals, where quantiles are a prominent object.

[7] A total preorder is a binary relation on $\mathcal{X}$ satisfying, for all $X, Y, Z \in \mathcal{X}$, (i) $X \preceq Y$ and $Y \preceq Z$ imply $X \preceq Z$, and (ii) $X \preceq Y$ or $Y \preceq X$ holds.

[8] That is, $X \preceq Y$ if and only if $\mathcal{R}(X) \leqslant \mathcal{R}(Y)$.

if $\preceq$ can be numerically represented by a constant times a quantile. More precisely, $\preceq$ being quantile maximizing means that it can be represented either by $\mathcal{R} = \lambda Q_p^L$ for some $p \in (0,1]$ and $\lambda \in \mathbb{R}$, or by $\mathcal{R} = \lambda Q_p^R$ for some $p \in [0,1)$ and $\lambda \in \mathbb{R}$. Note that λ may be 0 (leading to a degenerate $\preceq$) or negative (which may be seen as a quantile minimizer).

Theorem 5. *A law-based total preorder $\preceq$ on $\mathcal{X}$ with certainty equivalents is $\mathcal{G}^*$-invariant if and only if it is quantile maximizing.*

A sketch of the proof of Theorem 5 is as follows. Certainty equivalents and $\mathcal{G}^*$-invariance guarantee that the preorder $\preceq$ has a numerical representation $\mathcal{R}$. Let $h(c) = \mathcal{R}(c)$ for $c \in \mathbb{R}$. There are two possible cases: i) If h is strictly monotone, then $h_L^{-1} \circ \mathcal{R}$ is $\mathcal{G}^*$-ordinal. It follows from Theorem 1 that $h_L^{-1} \circ \mathcal{R}$ is a left or right quantile, which further implies $\mathcal{R}$ is a strictly monotone transform of a quantile. ii) If h is a constant, then $\mathcal{R}$ is a constant. See the details of the proof in Appendix A. The existence of certainty equivalents is needed for Theorem 5 to ensure the existence of a numerical representation (otherwise, a counter-example can be constructed).

To distinguish the left and right quantiles for quantile maximizing preferences characterized in Theorem 5, we need to enhance $\mathcal{G}^*$-invariance as in the following proposition, in a way similar to Proposition 1.

Proposition 6. *For a law-based total preorder $\preceq$ on $\mathcal{X}$ with certainty equivalents,*

(i) it is $\mathcal{G}^L$-invariant if and only if it is represented by λQ_p^L for some $p \in (0,1]$ and $\lambda \in \mathbb{R}$;

(ii) it is $\mathcal{G}^R$-invariant if and only if it is represented by λQ_p^R for some $p \in [0,1)$ and $\lambda \in \mathbb{R}$.

Other generalizations, such as those on domains of continuously distributed random variables or characterizing medians, can be done analogously using other results from Sections 3 and 4. We omit a detailed discussion.

6 Conclusion

We found a minimalist characterization of quantiles on a general set of distributions using only one axiom of ordinality. This result leads to axiomatizations of the median and the median interval on different domains of distributions, as well as an axiomatization of quantile maximization, each with a minimal number of axioms. As far as we are aware, this paper contains the first characterization of quantiles in the literature that does not impose monotonicity. Our main results are summarized in Table 2 below.

domain $\mathcal{M}$	$\mathcal{G}$-ordinality	$\rho : \mathcal{M} \to \mathbb{R}$	result
$\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$	$\mathcal{G}^*$ (increasing continuous)	all quantiles	Theorem 2
	$\mathcal{G}^L$ (increasing left-continuous)	left quantiles	Proposition 1
	$\mathcal{G}^R$ (increasing right-continuous)	right quantiles	
$\mathcal{M}_c^* \subseteq \mathcal{M} \subseteq \mathcal{M}_0^*$	$\mathcal{G}^*$ (increasing continuous)	all quantiles	Proposition 2
$\mathcal{M}_c^\diamond \subseteq \mathcal{M} \subseteq \mathcal{M}_0^\diamond$	$\mathcal{G}^\diamond$ (strictly increasing continuous)		
$\mathcal{M}_c^* \subseteq \mathcal{M} \subseteq \mathcal{M}_0^*$	$\mathcal{G}_\pm^*$ (monotone continuous)	the median	Theorem 3
$\mathcal{M}_c^\diamond \subseteq \mathcal{M} \subseteq \mathcal{M}_0^\diamond$	$\mathcal{G}_\pm^\diamond$ (strictly monotone continuous)		

domain $\mathcal{M}$	$\mathcal{G}$-ordinality	$\rho : \mathcal{M} \to \mathbb{I}$	result
$\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$	$\mathcal{G}^*$ (increasing continuous)	quantile intervals	Proposition 4
	$\mathcal{G}_\pm^*$ (monotone continuous)	equal-tailed intervals	
	minimal $\mathcal{G}^*$-ordinality	quantile singletons	Theorem 4
	minimal $\mathcal{G}_\pm^*$-ordinality	median intervals	

Table 2: Summary of characterization results

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Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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A Proofs of all results

A.1 Proof of results in Section 3

We first present proofs for the results in Section 3 which characterize real-valued mappings $\rho : \mathcal{M} \to \mathbb{R}$ that satisfy $\mathcal{G}$-ordinality for different choices of $\mathcal{G}$ and $\mathcal{M}$.

Proof of Theorem 2. We first show the "if" part. Note that for functions in $\mathcal{G}^*$, $T^{[\phi]}(F)(x) = F(\phi_R^{-1}(x))$, $x \in \mathbb{R}$, where $\phi_R^{-1}(x) = \inf\{y \in \mathbb{R} : \phi(y) > x\}$ with the convention $\inf \emptyset = \infty$. If ρ is a left quantile, then there exists $p \in (0, 1]$ such that $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}$. Hence for $\phi \in \mathcal{G}^*$, we have

$$\begin{aligned}
\rho(T^{[\phi]}(F)) &= \inf\{x \in \mathbb{R} : F(\phi_R^{-1}(x)) \geqslant p\} \\
&= \inf\{x \in \mathbb{R} : \phi_R^{-1}(x) \geqslant F_L^{-1}(p)\} \\
&= \inf\{x \in \mathbb{R} : x \geqslant \phi(F_L^{-1}(p))\} = \phi(\rho(F)).
\end{aligned}$$

If ρ is a right quantile, then there exists $p \in [0, 1)$ such that $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}$. Hence for $\phi \in \mathcal{G}^*$, if $F(F_R^{-1}(p)) > p$, we have

$$\begin{aligned}
\rho(T^{[\phi]}(F)) &= \inf\{x \in \mathbb{R} : F(\phi_R^{-1}(x)) > p\} \\
&= \inf\{x \in \mathbb{R} : \phi_R^{-1}(x) \geqslant F_R^{-1}(p)\} \\
&= \inf\{x \in \mathbb{R} : x \geqslant \phi(F_R^{-1}(p))\} = \phi(\rho(F)).
\end{aligned}$$

If $F(F_R^{-1}(p)) = p$, similarly as above, we have

$$\rho(T^{[\phi]}(F)) = \inf\{x \in \mathbb{R} : \phi_R^{-1}(x) > F_R^{-1}(p)\} = \phi(F_R^{-1}(p)) = \phi(\rho(F)).$$

We next focus on the "only if" part. We will first show the result on $\mathcal{M}_c$ and then extend it to $\mathcal{M}$. For $F \in \mathcal{M}$, suppose F is flat over (a, b) with $a < b$. Let $\phi(x) = x\mathbb{1}_{\{x \notin (a,b)\}} + \left(a + \frac{(x-a)^2}{b-a}\right)\mathbb{1}_{(a,b)}(x)$. Then we have $\phi \in \mathcal{G}^*$ and $T^{[\phi]}(F) = F$. Using $\mathcal{G}^*$-ordinality, we have $\rho(F) = \rho(T^{[\phi]}(F)) = \phi(\rho(F))$. This implies $\rho(F) \notin (a, b)$. Hence for any $F \in \mathcal{M}$, there exists a constant p_F depending on F such that

$$\rho(F) = F_L^{-1}(p_F) \text{ for } p_F \in (0, 1], \text{ or } \rho(F) = F_R^{-1}(p_F) \text{ for } p_F \in [0, 1). \tag{5}$$

For $F \in \mathcal{M}_c^*$, let

$$\phi(x) = \begin{cases} F_R^{-1}(0), & x \leqslant 0 \\ F_L^{-1}(x), & 0 < x \leqslant 1 \\ F_L^{-1}(1), & x > 1. \end{cases} \tag{6}$$

Then we have $\phi \in \mathcal{G}^*$ and $T^{[\phi]}(F_U) = F$, where F_U is uniform on $[0, 1]$. Hence,

$$\rho(F) = \rho(T^{[\phi]}(F_U)) = \phi(\rho(F_U)). \tag{7}$$

By (5), we have $\rho(F_U) \in [0, 1]$. Hence combination of (6) and (7) yields that for $F \in \mathcal{M}_c^*$, with $p = \rho(F_U)$,

$$\rho(F) = \begin{cases} F_L^{-1}(p), & p \in (0, 1] \\ F_R^{-1}(0), & p = 0. \end{cases} \tag{8}$$

Let $\mathcal{M}_c^I$ denote the set of all continuous distributions in $\mathcal{M}_c$. Then for $F \in \mathcal{M}_c^I$, we have $F \in \mathcal{G}^*$ and $T^{[F]}(F) = F_U$. It follows from $\mathcal{G}^*$-ordinality that $\rho(T^{[F]}(F)) = F(\rho(F))$, which together with $\rho(T^{[F]}(F)) = \rho(F_U) = p$ yields $F(\rho(F)) = p$. Note that $\rho(F) \notin (F_L^{-1}(p), F_R^{-1}(p))$. Hence we have $\rho(F) = F_L^{-1}(p)$ for $p \in (0, 1]$ or $\rho(F) = F_R^{-1}(p)$ for $p \in [0, 1)$.

Next we show that $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}_c^I$ or $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}_c^I$. This is obvious for $p = 0$ or $p = 1$. We next consider the case $0 < p < 1$. We assume by contradiction that there exist $F, G \in \mathcal{M}_c^I$ satisfying $F_L^{-1}(p) < F_R^{-1}(p)$ and $G_L^{-1}(p) < G_R^{-1}(p)$

such that $\rho(F) = F_L^{-1}(p)$ and $\rho(G) = G_R^{-1}(p)$. Let

$$\phi(x) = \frac{x - F_L^{-1}(p)}{F_R^{-1}(p) - F_L^{-1}(p)} \mathbb{1}_{\{F_L^{-1}(p) < x \leqslant F_R^{-1}(p)\}} + \mathbb{1}_{\{x > F_R^{-1}(p)\}},$$

and

$$\widehat{\phi}(x) = \frac{x - G_L^{-1}(p)}{G_R^{-1}(p) - G_L^{-1}(p)} \mathbb{1}_{\{G_L^{-1}(p) < x \leqslant G_R^{-1}(p)\}} + \mathbb{1}_{\{x > G_R^{-1}(p)\}}.$$

Note that $\phi, \widehat{\phi} \in \mathcal{G}^*$ and $T^{[\phi]}(F) = T^{[\widehat{\phi}]}(G)$. Using $\mathcal{G}^*$-ordinality, we have

$$\rho(T^{[\phi]}(F)) = \phi(\rho(F)) = 0 \text{ and } \rho(T^{[\widehat{\phi}]}(G)) = \widehat{\phi}(\rho(G)) = 1,$$

leading to a contradiction. Hence we have $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}_c^I$ or $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}_c^I$.

Without loss of generality, we assume $p \in (0, 1]$ and $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}_c^I$. Therefore, we have

$$\rho(F) = F_L^{-1}(p) \text{ for all } F \in \mathcal{M}_c^* \cup \mathcal{M}_c^I \text{ with } p \in (0, 1]. \tag{9}$$

We next show that (9) holds for all $F \in \mathcal{M}_c$. We assume by contradiction that there exists $F \in \mathcal{M}_c \setminus (\mathcal{M}_c^* \cup \mathcal{M}_c^I)$ such that $\rho(F) \neq F_L^{-1}(p)$. Note that by (5), there exists $p_F \in [0, 1]$ such that $\rho(F) = F_L^{-1}(p_F)$ or $\rho(F) = F_R^{-1}(p_F)$.

We first consider the case $\rho(F) = F_L^{-1}(p_F)$. This implies that $p_F \neq p$ and $F_L^{-1}(p_F) \neq F_L^{-1}(p)$. Without loss of generality, we assume $p_F < p$ and $F_L^{-1}(p_F) < F_L^{-1}(p)$. We distinguish two scenarios: $F_L^{-1}(t)$ is continuous over $[p_F, p)$ and $F_L^{-1}(t)$ is not continuous at some $t_0 \in [p_F, p)$.

If $F_L^{-1}(t)$ is continuous over $[p_F, p)$, let

$$\phi(x) = F_L^{-1}(p_F) \mathbb{1}_{\{x \leqslant F_L^{-1}(p_F)\}} + x \mathbb{1}_{\{F_L^{-1}(p_F) < x < F_L^{-1}(p)\}} + F_L^{-1}(p) \mathbb{1}_{\{x \geqslant F_L^{-1}(p)\}},$$

implying that $\phi \in \mathcal{G}^*$ and $T^{[\phi]}(F) \in \mathcal{M}_c^*$. Hence in light of (9), $\rho(T^{[\phi]}(F)) = (T^{[\phi]}(F))_L^{-1}(p) = \phi(F_L^{-1}(p)) = F_L^{-1}(p)$. However, using $\mathcal{G}^*$-ordinality, we have

$$\rho(T^{[\phi]}(F)) = \phi(\rho(F)) = \phi(F_L^{-1}(p_F)) = F_L^{-1}(p_F) < F_L^{-1}(p),$$

leading to a contradiction.

If $F_L^{-1}(t)$ is not continuous at $t_0 \in [p_F, p)$, let

$$\phi(x) = (x - F_L^{-1}(t_0)) \mathbb{1}_{\{F_L^{-1}(t_0) < x \leqslant F_R^{-1}(t_0)\}} + (F_R^{-1}(t_0) - F_L^{-1}(t_0)) \mathbb{1}_{\{x > F_R^{-1}(t_0)\}}.$$

Then it follows that $\phi \in \mathcal{G}^*$. By $\mathcal{G}^*$-ordinality, we have $\rho(T^{[\phi]}(F)) = \phi(\rho(F)) = \phi(F_L^{-1}(p_F)) = 0$. Moreover, define $G \in \mathcal{M}_c$ via $G_L^{-1}(t) = (t - t_0 + F_L^{-1}(t_0))\mathbb{1}_{\{0<t\leqslant t_0\}} + (t + F_R^{-1}(t_0))\mathbb{1}_{\{t_0<t\leqslant 1\}}$. Note that $G \in \mathcal{M}_c^I$ and $T^{[\phi]}(G) = T^{[\phi]}(F)$. However, by $\mathcal{G}^*$-ordinality, $\rho(T^{[\phi]}(G)) = \phi(\rho(G)) = \phi(G_L^{-1}(p)) = F_R^{-1}(t_0) - F_L^{-1}(t_0) > 0 = \rho(T^{[\phi]}(F))$, leading to a contradiction.

We next focus on the case $\rho(F) = F_R^{-1}(p_F)$. If $p_F \neq p$, the proof follows analogously as the proof of the case $\rho(F) = F_L^{-1}(p_F)$. We next consider the case $p_F = p < 1$ which implies $F_L^{-1}(p) < F_R^{-1}(p) = \rho(F)$. Let $\phi(x) = (x - F_L^{-1}(p))\mathbb{1}_{\{F_L^{-1}(p)<x\leqslant F_R^{-1}(p)\}} + (F_R^{-1}(p) - F_L^{-1}(p))\mathbb{1}_{\{x>F_R^{-1}(p)\}}$. Then we have $\phi \in \mathcal{G}^*$. By $\mathcal{G}^*$-ordinality, we have $\rho(T^{[\phi]}(F)) = \phi(\rho(F)) = F_R^{-1}(p) - F_L^{-1}(p)$. Similarly as in the proof of case $\rho(F) = F_L^{-1}(p_F)$, define $G \in \mathcal{M}_c^I$ by $G_L^{-1}(t) = (t - p + F_L^{-1}(p))\mathbb{1}_{\{0<t\leqslant p\}} + (t + F_R^{-1}(p))\mathbb{1}_{\{p<t\leqslant 1\}}$. It follows that $T^{[\phi]}(G) = T^{[\phi]}(F)$. By $\mathcal{G}^*$-ordinality, $\rho(T^{[\phi]}(G)) = \phi(\rho(G)) = \phi(G_L^{-1}(p)) = \phi(F_L^{-1}(p)) = 0$, leading to a contradiction. Therefore, we have $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}_c \setminus (\mathcal{M}_c^* \cup \mathcal{M}_c^I)$, which further implies

$$\rho(F) = F_L^{-1}(p) \text{ for all } F \in \mathcal{M}_c. \tag{10}$$

We next extend (10) from $\mathcal{M}_c$ to $\mathcal{M}$. Let $\phi(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$, $x \in \mathbb{R}$. For $F \in \mathcal{M} \setminus \mathcal{M}_c$, $T^{[\phi]}(F) \in \mathcal{M}_c$. It follows from (10) that $\rho(T^{[\phi]}(F)) = (T^{[\phi]}(F))_L^{-1}(p) = \phi(F_L^{-1}(p))$. Moreover, by $\mathcal{G}^*$-ordinality, we have $\rho(T^{[\phi]}(F)) = \phi(\rho(F))$. Since ϕ is strictly increasing, we have $\rho(F) = F_L^{-1}(p)$. Therefore, ρ is a left quantile.

If (9) is changed to $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}_c^* \cup \mathcal{M}_c^I$ with $p \in [0, 1)$, we can similarly show that ρ is a right quantile on $\mathcal{M}$. This completes the proof. $\square$

Proof of Proposition 1. The "if" parts of both (i) and (ii) follow similarly as in the proof of Theorem 2. We next focus on the "only if" parts.

(i) By Theorem 2, ρ is a left or right quantile. We next exclude the right quantile. We assume by contradiction that there exists $p \in [0, 1)$ such that $\rho(F) = F_R^{-1}(p)$ for all $F \in \mathcal{M}$. Let $\phi(x) = -\mathbb{1}_{\{x\leqslant 0\}} + p\mathbb{1}_{\{x>p\}}$ and F_U be uniform on $[0, 1]$. Then we have $\phi \in \mathcal{G}^L$. By $\mathcal{G}^L$-ordinality, we have $\rho(T^{[\phi]}(F_U)) = \phi(\rho(F_U)) = \phi(p)$. Moreover, direct calculation gives $\rho(T^{[\phi]}(F_U)) = (T^{[\phi]}(F_U))_R^{-1}(p) = p \neq \phi(p)$ for $p \in [0, 1)$, leading to a contradiction. Hence ρ is a left quantile.

(ii) Define $\widehat{\rho} : \mathcal{M}_c \to \mathcal{M}_c$ by $\widehat{\rho} = -\rho \circ T^{[\psi]}$, where $\psi(x) = -x$ is the negative identity function. Then one can check that $\widehat{\rho}$ satisfies $\mathcal{G}^L$-ordinality. It follows from (i) that $\widehat{\rho}$ is a left quantile, implying that $\rho = -\widehat{\rho} \circ T^{[\psi]}$ is a right quantile. $\square$

Proof of Proposition 2. The "if" parts of both (i) and (ii) follow similarly as in the proof of Theorem 2. The "only if" part of (i) is implied by (8), which shows that for $F \in \mathcal{M}_c^*$, $\rho(F)$ is a p-quantile of F. The rest follows from the same arguments in Theorem 2.

Next we show the "only if" part of (ii). For $F \in \mathcal{M}_c^\diamond$, let

$$\phi(x) = \begin{cases} x + F_R^{-1}(0), & x \leqslant 0 \\ F_L^{-1}(x), & 0 < x \leqslant 1 \\ x - 1 + F_L^{-1}(1), & x > 1. \end{cases} \tag{11}$$

Then we have $\phi \in \mathcal{G}^\diamond$ and $T^{[\phi]}(F_U) = F$, where F_U is uniform on $[0,1]$. Hence,

$$\rho(F) = \rho(T^{[\phi]}(F_U)) = \phi(\rho(F_U)). \tag{12}$$

Using the same argument as in the proof of Theorem 2, we can show that $\rho(F) \notin (a,b)$ if F is flat over an interval (a,b). This implies $\rho(F_U) \in [0,1]$ as F_U is flat over $(-\infty, 0)$ and $(1, \infty)$. Analogously as in the proof of Theorem 2, we can extend our conclusion on $\mathcal{M}_c^\diamond$ to any $\mathcal{M}$ such that $\mathcal{M}_c^\diamond \subseteq \mathcal{M} \subseteq \mathcal{M}_0^\diamond$. $\square$

Proof of Theorem 3. (i) The "if" part is straightforward to check. To show the "only if" part, by Proposition 2, we know that ρ is a quantile, and denote by p its level. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = -x$ which is the negative identity function. Since $\psi \in \mathcal{G}_{\pm}^*$, we have for $F \in \mathcal{M}_c^*$, $-F_L^{-1}(p) = \psi(\rho(F)) = \rho(T^{[\psi]}(F)) = (T^{[\psi]}(F))_L^{-1}(p) = -F_L^{-1}(1-p)$. Hence $p = 1/2$ and ρ is the median. (ii) follows from the same argument. $\square$

Proof of Proposition 3. The "if" part can be checked directly. We next focus on the "only if" part. For $m > 0$, let $t_m : x \mapsto \min(\max(-m, x), m)$, which is the truncation function at $-m$ and m.

We first assume $\mathcal{M} = \mathcal{M}_c$. Note that a combination of (i) and (ii) yields that $\rho(T^{[\phi]}(F)) = \phi(\rho(F))$ holds for all piece-wise linear (with finite pieces) and continuously increasing functions ϕ. Using $F \in \mathcal{M}_c$, we have for $\phi \in \mathcal{G}^*$, $T^{[\phi]}(F) = T^{[\phi^*]}(F)$, where $\phi^*(x) = \phi(t_m(x))$ with $F(m) = 1$ and $F(-m) = 0$. Since ϕ^* is increasing and continuous on $[-m, m]$, there exists a sequence of piece-wise linear (with finite pieces) and continuously increasing functions ϕ_n such that $\sup_{x \in \mathbb{R}} |\phi_n(x) - \phi^*(x)| \to 0$ as $n \to \infty$. Hence using the continuity of ρ, we have

$$\rho(T^{[\phi]}(F)) = \rho(T^{[\phi^*]}(F)) = \lim_{n\to\infty} \rho(T^{[\phi_n]}(F)) = \lim_{n\to\infty} \phi_n(\rho(F)) = \phi^*(\rho(F)).$$

Setting $m > |\rho(F)|$, we have $\rho(T^{[\phi]}(F)) = \phi^*(\rho(F)) = \phi(\rho(F))$ for all $\phi \in \mathcal{G}^*$. By Theorem 2, ρ is a left or right quantile on $\mathcal{M}_c$.

We next extend the result from $\mathcal{M}_c$ to $\mathcal{M}$. Without loss of generality, assume that for some $p \in (0,1]$, $\rho(F) = F_L^{-1}(p)$ for all $F \in \mathcal{M}_c$. Using the above arguments again, for $F \in \mathcal{M}$, $\rho(T^{[t_m]}(F)) = t_m(\rho(F))$ holds for $m > 0$. This implies that for $F \in \mathcal{M}$, $t_m(\rho(F)) =$

$\rho(T^{[t_m]}(F)) = (T^{[t_m]}(F))_L^{-1}(p) = t_m(F_L^{-1}(p))$. Setting $m > |F_L^{-1}(p)|$, we have $\rho(F) = F_L^{-1}(p)$. This completes the proof. $\square$

A.2 Proof of results in Section 4

Next we present the proofs of results in Section 4 which characterize interval-valued mappings $\rho : \mathcal{M} \to \mathbb{I}$ satisfying $\mathcal{G}$-ordinality for different choices of $\mathcal{G}$ and $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$.

Proof of Proposition 4. The "if" parts of both (i) and (ii) are straightforward. We next consider the "only if" parts.

(i) We write $\rho = [\rho_1, \rho_2]$. By Theorem 2, there exist $p_1, p_2 \in [0,1]$ with $p_1 \leqslant p_2$ such that $\rho_1(F) = F_L^{-1}(p_1)$ or $F_R^{-1}(p_1)$ and $\rho_2(F) = F_L^{-1}(p_2)$ or $F_R^{-1}(p_2)$ for all $F \in \mathcal{M}$. This implies the four scenarios (a)-(d) of quantile intervals given in Section 4. We establish the claim of (i).

(ii) Using the conclusion of (i), $\mathcal{G}^*$-ordinality implies four scenarios: (a)-(d). We first consider scenario (a). Note that the negative identity function $\psi(x) = -x \in \mathcal{G}^*_\pm$. Hence $\mathcal{G}^*_\pm$-ordinality gives $\rho(T^{[\psi]}(F)) = \psi(\rho(F)) = [-F_R^{-1}(p_2), -F_L^{-1}(p_1)]$. Moreover, direct calculation shows

$$\rho(T^{[\psi]}(F)) = [(T^{[\psi]}(F))_L^{-1}(p_1), (T^{[\psi]}(F))_R^{-1}(p_2)] = [-F_R^{-1}(1-p_1), -F_L^{-1}(1-p_2)].$$

This implies that $p_1 + p_2 = 1$ and $p_1 \in (0, 1/2]$. Analogously, we can show that for scenario (b), $p_1 + p_2 = 1$ and $p_1 \in (0, 1/2)$.

We next focus on scenario (c). Note that $\mathcal{G}^*_\pm$-ordinality gives $\rho(T^{[\psi]}(F)) = \psi(\rho(F)) = [-F_L^{-1}(p_2), -F_L^{-1}(p_1)]$. Moreover, direct calculation shows

$$\rho(T^{[\psi]}(F)) = [(T^{[\psi]}(F))_L^{-1}(p_1), (T^{[\psi]}(F))_L^{-1}(p_2)] = [-F_R^{-1}(1-p_1), -F_R^{-1}(1-p_2)].$$

It is impossible that $F_L^{-1}(p_2) = F_R^{-1}(1-p_1)$ for all $F \in \mathcal{M}_c$, leading to a contradiction. Hence (c) does not satisfy $\mathcal{G}^*_\pm$-ordinality. Similarly, we can show that scenario (d) does not satisfy $\mathcal{G}^*_\pm$-ordinality. We establish the result in (ii). $\square$

Proof of Theorem 4. It follows from Proposition 4 and the definition of minimality. $\square$

Proof of Proposition 5. Since the p-quantile interval is elicitable on $\mathcal{M}_c$, and the singleton mappings $F \mapsto [F_L^{-1}(p), F_L^{-1}(p)]$ and $F \mapsto [F_R^{-1}(p), F_R^{-1}(p)]$ are not elicitable on $\mathcal{M}_c$ (see Kou and Peng (2016)), it suffices to verify that the other quantile intervals are not elicitable on $\mathcal{M}_c$. We look at the type $\rho(F) = [F_L^{-1}(p), F_R^{-1}(q)]$ for $p < q$, as the other three types follow from the same argument. Suppose that S is a score function for ρ. Let F and G be the uniform distributions on $[0,1]$ and $[\varepsilon, 1+\varepsilon]$, respectively, where $\varepsilon = (q-p)/2$. It is clear that $\rho(F) = [p, q]$

and $\rho(G) = [p+\varepsilon, q+\varepsilon]$. Using the definition of the score function, for $H = (F+G)/2$, we have

$$\arg\min_{x\in\mathbb{R}} \int_{\mathbb{R}} S(x,y)\mathrm{d}H(y) = [p,q] \cap [p+\varepsilon, q+\varepsilon] = [p+\varepsilon, q] \neq \rho(H),$$

a contradiction. Hence, both (i) and (ii) follow from the above argument and Proposition 4. $\square$

A.3 Proof of results in Section 5

Finally we present the proofs of Theorem 5 and Proposition 6 and some related technical discussions. In what follows, let $\mathcal{X}_b$ be the set of all bounded random variables.

Lemma 1. *(i) For a law-based and $\mathcal{G}^*$-invariant total preorder $\preceq$ on $\mathcal{X}$, it has a numerical representation if it has certainty equivalents;*

(ii) For a law-based and $\mathcal{G}^*$-invariant total preorder $\preceq$ on $\mathcal{X}_b$, it has a numerical representation if and only if it has certainty equivalents.

Proof. (i) We distinguish three scenarios: $0 \prec 1$, $0 \sim 1$ and $1 \prec 0$. We first assume $0 \prec 1$ and show that $x \prec y$ for any $x < y$. We assume by contradiction that there exist $x, y \in \mathbb{R}$ with $x < y$ such that $y \preceq x$. By $\mathcal{G}^*$-invariance, it follows that $\phi(y) \preceq \phi(x)$ for all $\phi \in \mathcal{G}^*$. Let $\phi \in \mathcal{G}^*$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Then we have $1 \preceq 0$, leading to a contradiction. Hence $x \prec y$ if $x < y$.

Let $\mathcal{X}_1$ be the subset of $\mathcal{X}$ consisting of all rational numbers. By the property of having certainty equivalents (this property will be abbreviated as CE hereafter), for any $X, Y \in \mathcal{X}$ satisfying $X \prec Y$, there exist $x, y \in \mathbb{R}$ such that $X \sim x$, $Y \sim y$ and $x < y$. For $c \in \mathcal{X}_1$ with $x < c < y$, it follows that $X \prec c \prec Y$. Using Theorem 2.6 of Föllmer and Schied (2016), the preorder $\preceq$ has a numerical representation. The case $1 \prec 0$ can be shown similarly.

We next focus on the case $0 \sim 1$. By $\mathcal{G}^*$-invariance, $x \sim y$ for all $x, y \in \mathbb{R}$. It follows from CE that $X \sim Y$ for all $X, Y \in \mathcal{X}$. Hence the preorder $\preceq$ has a numerical representation. This completes the proof of (i).

(ii) The "if" part follows from (i). We next consider the "only if" part. We distinguish three scenarios: $0 \prec 1$, $0 \sim 1$ and $1 \prec 0$. We start with $0 \prec 1$, implying $x \prec y$ if $x < y$. For $Y \in \mathcal{X}_b$, there exist $m_1, m_2 \in \mathbb{R}$ such that $m_1 \leqslant Y \leqslant m_2$. We next show that $m_1 - 1 \preceq Y \preceq m_2 + 1$. We assume by way of contradiction that $m_2 + 1 \prec Y$. Let $\phi(x) = \max(x - m_2, 0)$. Note that $\phi \in \mathcal{G}^*$. By $\mathcal{G}^*$-invariance, we have $1 = \phi(m_2+1) \preceq \phi(Y) = 0$, leading to a contradiction. Hence $Y \preceq m_2 + 1$. We can similarly show $m_1 - 1 \preceq Y$. Next, assume by way of contradiction that there exists $X \in \mathcal{X}_b$ such that $X \prec x$ or $x \prec X$ for all $x \in \mathbb{R}$. It follows that there exists $c \in \mathbb{R}$ such that $x \prec X \prec y$ for $x < c < y$. Without loss of generality, we assume $X \prec c$. For $t \in \mathbb{R}$, it follows that $x \preceq X + t \prec c + t$ for all $x < c + t$. We denote $\mathcal{X}_2$ by a subset of $\mathcal{X}_b$ that is order dense in $\mathcal{X}_b$. Hence for $t \in \mathbb{R}$, there exists $Y_t \in \mathcal{X}_2$ such that $X + t \preceq Y_t \preceq c + t$. Note that if $t_1 < t_2$, $Y_{t_1} \preceq c + t_1 \prec c + \frac{t_1+t_2}{2} \prec Y_{t_2}$. Hence $\mathcal{X}_2$ is uncountable. By Theorem 2.6 of Föllmer and Schied (2016), the preorder $\preceq$ does not have a numerical representation. The above arguments show that the preorder $\preceq$ has certainty equivalents. Analogously, we can establish the claim for $1 \prec 0$.

For the case $0 \sim 1$, we have $x \sim y$ for all $x, y \in \mathbb{R}$. Since for every $X \in \mathcal{X}$, there exist $x, y \in \mathbb{R}$ such that $x \preceq X \preceq y$, we have $X \sim x$. Hence, the preorder $\preceq$ has certainty equivalents. $\square$

We next consider mapping satisfying $\mathcal{G}^*$-invariance, showing that $\mathcal{G}^*$-invariance is necessary and sufficient to ensure that $\mathcal{R}$ on $\mathcal{X}_b$ is a monotone function of quantile. However, an additional assumption is needed for mappings on general $\mathcal{X}$. For an increasing function ϕ, $\phi_L^{-1}(x) = \inf\{y \in \mathbb{R} : \phi(y) \geqslant x\}$ with the convention $\inf \emptyset = \infty$.

Recall that $\mathcal{R}$ represents the quantile maximizing preference $\preceq$ if and only if there exists a function h on $\mathbb{R}$ such that $\mathcal{R} = h \circ Q_p^L$ for $p \in (0, 1]$ or $\mathcal{R} = h \circ Q_p^R$ for $p \in [0, 1)$, where h is either strictly monotone or a constant (and it suffices to consider linear h as in Section 5). In what follows, $\mathcal{R}(A) = \{\mathcal{R}(X) : X \in A\}$ for a set $A \subseteq \mathcal{X}$. We treat the domains of $\mathcal{X}_b$ and a general $\mathcal{X}$ separately as the latter requires an additional assumption.

Lemma 2. *(i) A law-based mapping $\mathcal{R} : \mathcal{X}_b \to \mathbb{R}$ is $\mathcal{G}^*$-invariant if and only if it represents a quantile maximizing preference.*

(ii) A law-based mapping $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ satisfying $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathbb{R})$ is $\mathcal{G}^*$-invariant if and only if it represents a quantile maximizing preference.

Proof. The "if" part is obvious in both statements. We will show the "only if" part. Note that the difference between (i) and (ii) is that the condition $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathbb{R})$ is assumed in (ii). Below, the only places where we use the fact that $\mathcal{X}_b$ contains only bounded random variables are step 2 and step 5 (a), and the other steps can be unified for both cases. Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(c) = \mathcal{R}(c)$, $c \in \mathbb{R}$.

1. We first assume $\mathcal{R}(0) < \mathcal{R}(1)$. We claim that h is strictly increasing. We assume by way of contradiction that there exist $x, y \in \mathbb{R}$ with $x < y$ such that $h(x) \geqslant h(y)$. By $\mathcal{G}^*$-invariance, it follows that $h \circ \phi(x) \geqslant h \circ \phi(y)$ for all $\phi \in \mathcal{G}^*$. Let $\phi \in \mathcal{G}^*$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Then we have $h(0) \geqslant h(1)$, which contradicts with $\mathcal{R}(0) < \mathcal{R}(1)$. Hence h is strictly increasing on $\mathbb{R}$.

2. This step is only needed for the case $\mathcal{X}_b$ in (i). We claim that for $X \in \mathcal{X}_b$, there exist $c_1, c_2 \in \mathbb{R}$ with $c_1 \leqslant c_2$ such that $h(c_1) \leqslant \mathcal{R}(X) \leqslant h(c_2)$. Let $m_1, m_2 \in \mathbb{R}$ such that $m_1 \leqslant X \leqslant m_2$.

We assume by contradiction that $\mathcal{R}(X) \geqslant \sup_{c\in\mathbb{R}} h(c)$. Let $\phi(x) = \max(x - m_2, 0)$. Then $\phi(X) = 0$. By $\mathcal{G}^*$-invariance, we have $\mathcal{R}(\phi(X)) = \mathcal{R}(0) = h(0) \geqslant \sup_{c\in\mathbb{R}} h(\phi(c))$, which contradicts with the fact that h is strictly increasing. Hence we have $\mathcal{R}(X) \leqslant h(c_2)$ for some $c_2 \in \mathbb{R}$. Analogously, we can show that there exists $c_1 \in \mathbb{R}$ such that $h(c_1) \leqslant \mathcal{R}(X)$ and $c_1 \leqslant c_2$ is implied by the strict increasing monotonicity of h.

We next show that $\mathcal{R}(\mathcal{X}_b) = \mathcal{R}(\mathbb{R})$. If $\mathcal{R}(X) \notin \mathcal{R}(\mathbb{R})$, there exists $c_0 \in \mathbb{R}$ such that $h(c_0-) \leqslant \mathcal{R}(X) \leqslant h(c_0+)$, where $h(x-)$ and $h(x+)$ are the left and right limits, respectively, of h at a point x. Let $\phi(x) = x - c_0 + x_0$, where x_0 is a continuous point of h. Then $\phi \in \mathcal{G}^*$. Using $\mathcal{G}^*$-invariance, we have $h(\phi(c_0)-) \leqslant \mathcal{R}(\phi(X)) \leqslant h(\phi(c_0)+)$. Note that $h(\phi(c_0)-) = h(\phi(c_0)+) = h(x_0)$. Hence, $\mathcal{R}(\phi(X)) = h(x_0)$. Noting that $\phi_L^{-1} : x \mapsto x + c_0 - x_0 \in \mathcal{G}^*$, it follows from $\mathcal{G}^*$-invariance that $\mathcal{R}(X) = h(\phi_L^{-1}(x_0))$, leading to a contradiction. Hence, $\mathcal{R}(X) \in \mathcal{R}(\mathbb{R})$ for all $X \in \mathcal{X}_b$.

3. Using the above facts, by defining the mapping $\mathcal{R}'(X) = h_L^{-1}(\mathcal{R}(X))$ for $X \in \mathcal{X}$, we can verify that $\mathcal{R}'$ satisfies

$$\mathcal{R}'(c) = h_L^{-1}(h(c)) = c \text{ for all } c \in \mathbb{R}, \tag{13}$$

and $\mathcal{R}'$ is $\mathcal{G}^*$-invariant. Hence, for $\phi \in \mathcal{G}^*$ and $X, Y \in \mathcal{X}$, we have

$$\mathcal{R}'(X) = \mathcal{R}'(Y) \implies \mathcal{R}'(\phi(X)) = \mathcal{R}'(\phi(Y)).$$

Moreover, by (13), we have $\mathcal{R}'(\mathcal{R}'(X)) = \mathcal{R}'(X)$ for $X \in \mathcal{X}$. Hence, for $\phi \in \mathcal{G}^*$ and $X \in \mathcal{X}$, we have $\mathcal{R}'(\phi(\mathcal{R}'(X))) = \mathcal{R}'(\phi(X))$. This implies $\phi(\mathcal{R}'(X)) = \mathcal{R}'(\phi(X))$ for all $\phi \in \mathcal{G}^*$. In light of Theorem 1, we have $\mathcal{R}' = Q_p^L$ for $p \in (0, 1]$ or $\mathcal{R}' = Q_p^R$ for $p \in [0, 1)$.

Using $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathbb{R})$ and the fact that h is strictly increasing, we have $\mathcal{R} = h \circ \mathcal{R}'$. We establish the claim for the case $\mathcal{R}(0) < \mathcal{R}(1)$.

4. For the case $\mathcal{R}(0) > \mathcal{R}(1)$, let $\widetilde{\mathcal{R}} = -\mathcal{R}$. Then $\widetilde{\mathcal{R}}$ is law-based, $\mathcal{G}^*$-invariant and $\widetilde{\mathcal{R}}(0) < \widetilde{\mathcal{R}}(1)$. By the above arguments, there exists a strictly increasing h such that $\widetilde{\mathcal{R}} = h \circ Q_p^L$ for $p \in (0, 1]$ or $\widetilde{\mathcal{R}} = h \circ Q_p^R$ for $p \in [0, 1)$. Hence $\mathcal{R} = (-h) \circ Q_p^L$ for $p \in (0, 1]$ or $\mathcal{R} = (-h) \circ Q_p^R$ for $p \in [0, 1)$. Note that $-h$ is a strictly decreasing function.

5. Finally, we consider the case $\mathcal{R}(0) = \mathcal{R}(1)$. For $c_1 < c_2$, let $\phi(x) = c_1 + (c_2 - c_1)x$. Using $\mathcal{G}^*$-invariance, we have $\mathcal{R}(c_1) = \mathcal{R}(\phi(0)) = \mathcal{R}(\phi(1)) = \mathcal{R}(c_2)$. Hence $h \equiv h(0) \in \mathbb{R}$.

   (a) We first analyze the case of $\mathcal{X}_b$ in (i). If $\mathcal{R}(X) \neq h(0)$ for some $X \in \mathcal{X}_b$, without loss of generality, we assume $\mathcal{R}(X) < h(0)$. There exist $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2 \in \mathbb{R}$ such that $m_1 \leqslant X \leqslant m_2$. If $\mathcal{R}(X) \leqslant \mathcal{R}(X + m_2 - m_1)$, let $\phi(x) = \max(x - m_2 + m_1, m_1)$. Using $\mathcal{G}^*$-invariance, it follows that $\mathcal{R}(\phi(X)) \leqslant \mathcal{R}(\phi(X+m_2-m_1))$, implying $\mathcal{R}(m_1) \leqslant \mathcal{R}(X) < h(0) = \mathcal{R}(m_1)$, leading to a contradiction. If $\mathcal{R}(X) \geqslant \mathcal{R}(X + m_2 - m_1)$, let $\phi(x) = \min(x, m_2)$. Using $\mathcal{G}^*$-invariance, it follows that $\mathcal{R}(\phi(X)) \geqslant \mathcal{R}(\phi(X + m_2 - m_1))$, implying $\mathcal{R}(m_2) \leqslant \mathcal{R}(X) < h(0) = \mathcal{R}(m_2)$, leading to a contradiction. Hence, we have $\mathcal{R}(X) = h(0)$ and $\mathcal{R} \equiv h(0)$.

(b) In case of a general $\mathcal{X}$ in (ii), using the condition $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathbb{R})$ and $\mathcal{R}(\mathbb{R}) = \{h(0)\}$, we get $\mathcal{R}$ is a constant on $\mathcal{X}$.

Summarizing the above cases, we establish the desired result. □

Proof of Theorem 5. Theorem 5 follows directly from combining Lemmas 1 and 2. □

Proof of Proposition 6. (i) The "if" part is straightforward. We next consider the "only if" part. By Theorem 5, the preorder can be represented by $\mathcal{R} = \lambda Q_p^L$ for some $p \in (0, 1]$ and $\lambda \in \mathbb{R}$, or $\mathcal{R} = \lambda Q_p^R$ for some $p \in [0, 1)$ and $\lambda \in \mathbb{R}$. We next exclude the case $\mathcal{R} = \lambda Q_p^R$ for some $p \in [0, 1)$ and $\lambda \neq 0$. We assume for some $p \in [0, 1)$ and $\lambda \neq 0$, $X \preceq Y$ if and only if $\lambda Q_p^R(X) \leqslant \lambda Q_p^R(Y)$ for all $X, Y \in \mathcal{X}$. Let X be a random variable with distribution $p\delta_0 + (1-p)\delta_1$, where δ_x represents a probability mass distribution at x, and U be a uniform random variable distributed on $[0, 1]$. Hence we have $\lambda Q_p^R(X) = \lambda Q_p^R(U + 1 - p) = \lambda$, implying $X \sim U + 1 - p$. Note that $\phi(x) = \mathbb{1}_{\{x>1\}} \in \mathcal{G}^L$. It follows from $\mathcal{G}^L$-invariance that $\phi(X) \sim \phi(U + 1 - p)$. However, $0 = \lambda Q_p^R(\phi(X)) \neq \lambda Q_p^R(\phi(U+1-p)) = \lambda$, leading to a contradiction. Hence the case $\mathcal{R} = \lambda Q_p^R$ for some $p \in [0, 1)$ and $\lambda \neq 0$ does not satisfy $\mathcal{G}^L$-ordinality. We establish claim (i). (ii) follows from the same argument as in the proof of (i). □