# Eikosograms for teaching
# probabilistic independence and its modelling.

W.H. Cherry and R.W. Oldford
University of Waterloo

October 9, 2002

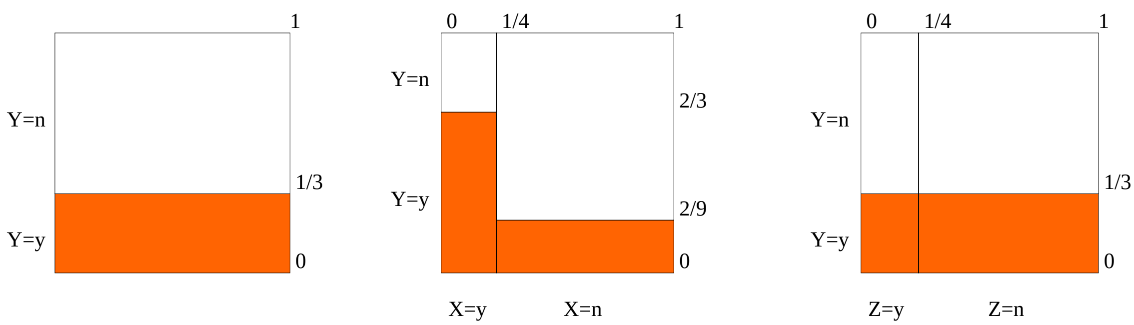
**Abstract**

This the paper where we look at the use of eikosograms to show independence.

## 1 Introduction

Teachers of probability have long used relative areas to teach probability. In Cherry and Oldford (2001) we argue for the use of a diagram, which we have called and eikosogram, to teach the fundamentals of probability. In this paper we employ the eikosogram to explore the topic of conditional independence.

## 2 Eikosograms




(a) Marginal of Y (b) Y depends on X (c) Y independent of Z

Figure 1: The eikosogram for one and for two variables.

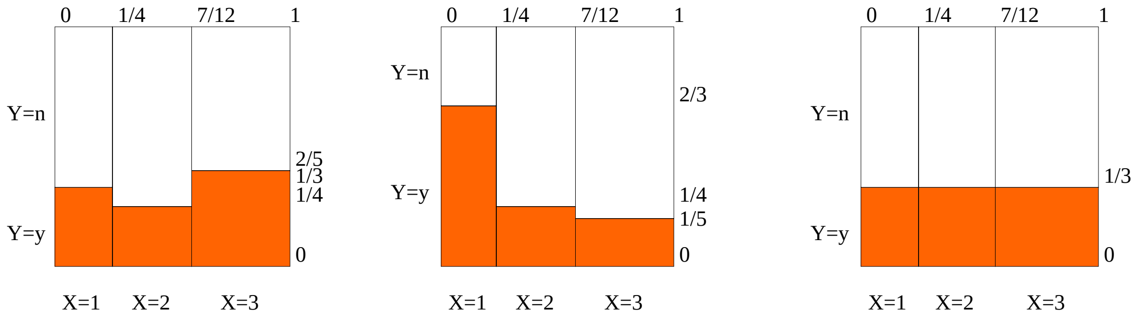
## 3 Observing independence

Must look at all three variables on the vertical axis.

There are four cases to consider

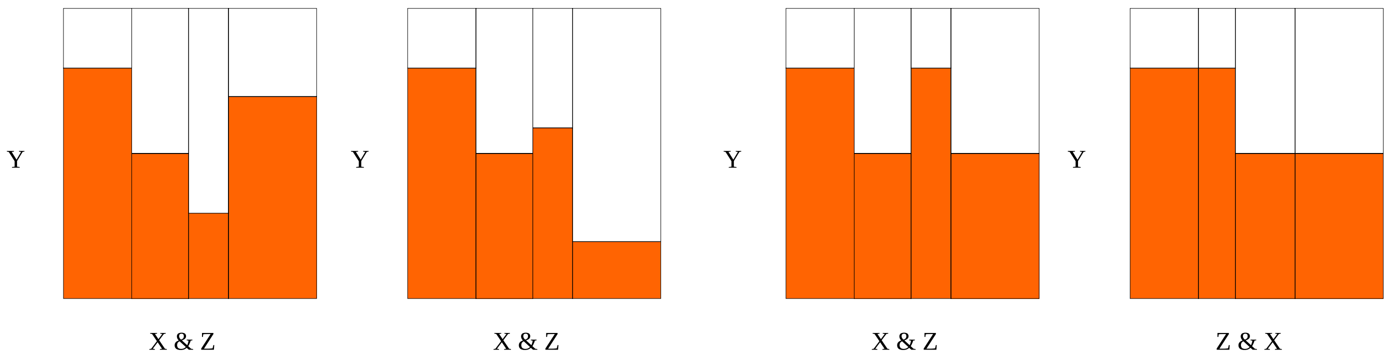
### 3.1 Case 1: All three diagrams are flat.

This is the case of mutual independence.




(a) Y depends on X (b) Pr(Y|X) declines with X (c) Y independent of X

Figure 2: Multiple categories for the conditioning variable.




(a) (b) (c) (d)

Figure 3: Various dependency relations (a) inconsistent dependence, (b) consistent dependence, (c) conditional independence, (d) as in (c) but $X$ & $Z$ are interchanged.

### 3.2 Case 2: one 4-flat, two (2,2)-flats

Note that it is **impossible** to have three 2-flats

### 3.3 Case 3: two (2,2)-flats, one no-flat

### 3.4 Case 4: three no-flats

## 4 Log-linear models

It is more common to parameterize the probability of the contingency table by the probabilities of the cells of the table with $p_{yxz}$ denoting the probability $Pr(Y = y, X = x, Z = z)$. In the examples considered above, each variable takes on only two values, for example $Y = \mathtt{y}$ and $Y = \mathtt{n}$, thereby yielding 8 cells having probabilities $p_{yxz}$. So as not to confuse the values of the variables with the variables themselves, we will now switch to more standard notation and have each binary variable take on the values 1 and 0 corresponding to the values $\mathtt{y}$ and $\mathtt{n}$, respectively. This means, for example, that

$$p_{010} = Pr(Y = 0, X = 1, Z = 0) = Pr(Y = \mathtt{n}, X = \mathtt{y}, Z = \mathtt{n})$$

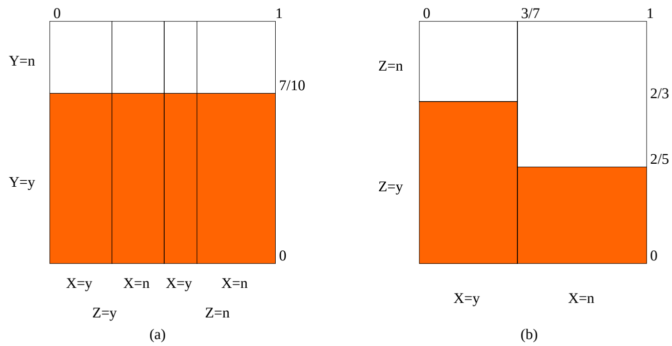



Figure 4: One 4-flat does not imply mutual independence of $Y$, $X$, and $Z$.

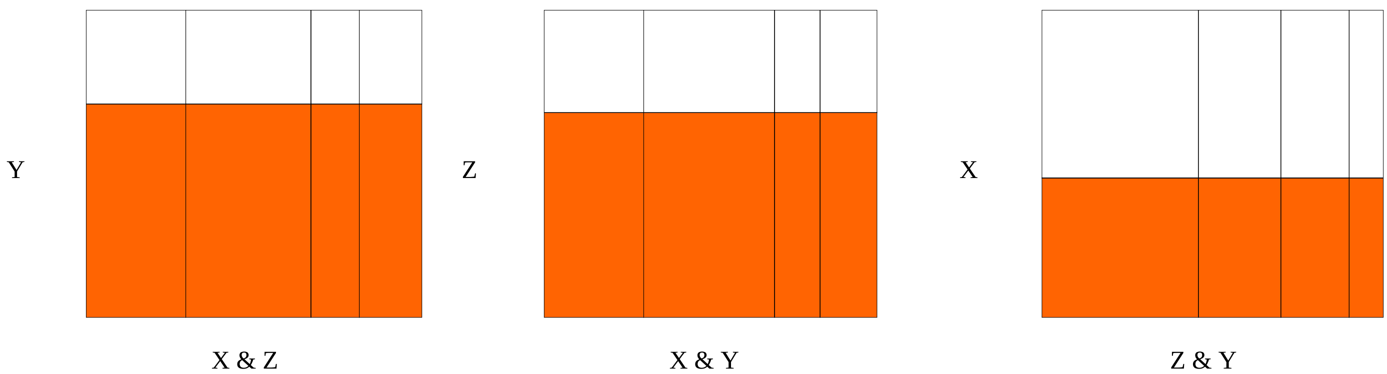



Figure 5: Mutual independence: any two 4-flats imply the third is a 4-flat; three 4-flats if and only if $Y$ $X$ and $Z$ are mutually independent.

In terms of our original notation:

$$\begin{array}{llcll} a = p_{011} + p_{111} & e = p_{111}/a & \text{or equivalently} & p_{111} = a \times e & p_{011} = a \times (1-e) \\ b = p_{001} + p_{101} & f = p_{101}/b & & p_{101} = b \times f & p_{001} = b \times (1-f) \\ c = p_{010} + p_{110} & g = p_{110}/c & & p_{111} = c \times g & p_{011} = c \times (1-g) \\ d = p_{000} + p_{100} & h = p_{100}/d & & p_{100} = d \times h & p_{000} = d \times (1-h) \end{array}$$

## 4.1 log-linear parameterization

For $y \in \{0,1\}$, $x \in \{0,1\}$, $z \in \{0,1\}$

$$p_{yxz} = p_{000}^{(1-y)(1-x)(1-z)} \; p_{001}^{(1-y)(1-x)z} \; p_{010}^{(1-y)x(1-z)} \; p_{011}^{(1-y)xz} \; p_{100}^{y(1-x)(1-z)} \; p_{101}^{y(1-x)z} \; p_{110}^{yx(1-z)} \; p_{111}^{yxz}$$

Taking logs and gathering like terms together we have

$$log(p_{yxz}) = u_0 + u_Y y + u_X x + u_Z z + u_{YX} yx + u_{YZ} yz + u_{XZ} xz + u_{YXZ} yxz$$

where the $u$s are simply the coefficients of the corresponding terms and so are functions of the $p_{yxz}$s. The complete set of equations are

$$log(p_{000}) \;\; = \;\; u_\phi$$

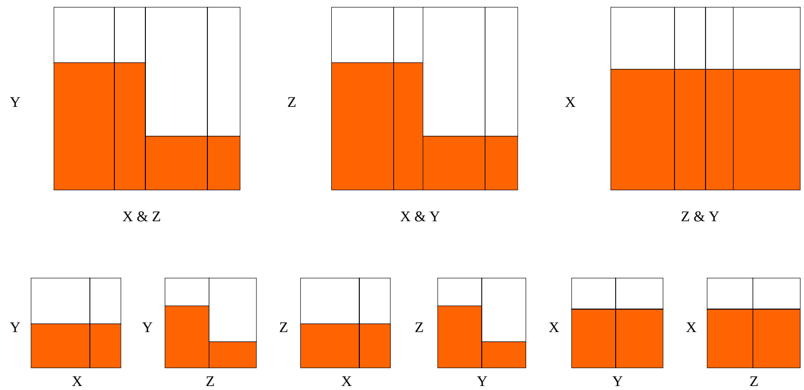



Figure 6: One 4-flat and two (2,2)-flats.

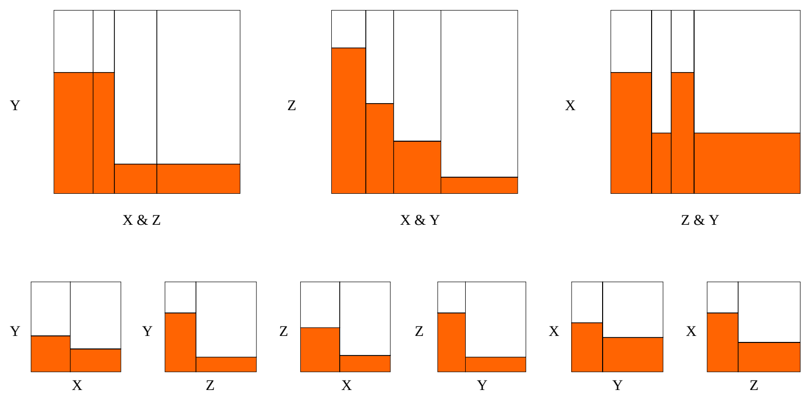



Figure 7: Two (2,2)-flats, one no-flat.

$$
\begin{aligned}
log(p_{100}) &= u_\phi + u_Y \\
log(p_{010}) &= u_\phi + \quad u_X \\
log(p_{001}) &= u_\phi + \quad\quad u_Z \\
log(p_{110}) &= u_\phi + u_Y + u_X + \quad u_{YX} \\
log(p_{101}) &= u_\phi + u_Y \quad + u_Z + \quad u_{YZ} \\
log(p_{011}) &= u_\phi + \quad u_X + u_Z + \quad u_{XZ} \\
log(p_{111}) &= u_\phi + u_Y + u_X + u_Z + u_{YX} + u_{YZ} + u_{XZ} + u_{YXZ}
\end{aligned}
$$

These so-called '$u$-terms' are the new parameters and are given the usual design interpretations with $u_Y$, $u_X$ and $u_Z$ each being the main effect of the corresponding variable, $u_{YX}$, $u_{YZ}$ and $u_{XZ}$ the two factor or first-order interaction terms, and $u_{YXZ}$ the three factor or second order interaction term.

It is easily seen that

$$
\begin{aligned}
u_\phi &= log(p_{000}) \\
u_Y &= log(p_{100}/p_{000}) \\
u_X &= log(p_{010}/p_{000}) \\
u_Z &= log(p_{001}/p_{000}) \\
u_{YX} &= log(p_{000}\ p_{110}/p_{010}\ p_{100})
\end{aligned}
$$

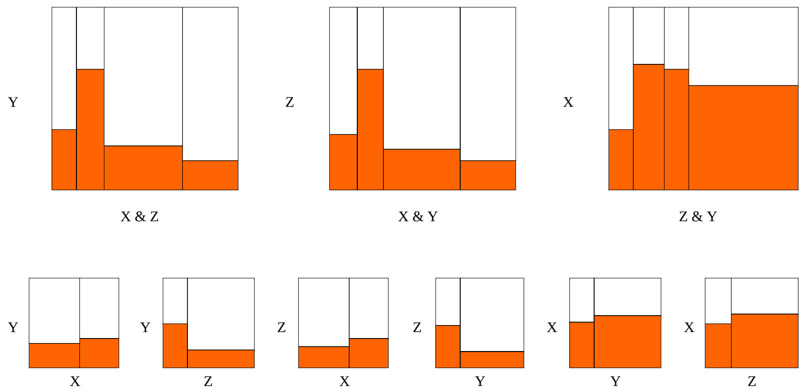



Figure 8: No flats. No independent variables.

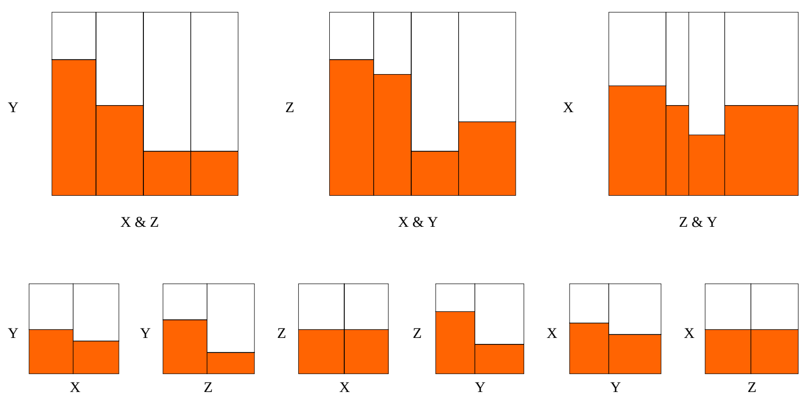



Figure 9: No flats. One marginal independence. No other independence

$$
\begin{aligned}
u_{YZ} &= log(p_{000}\ p_{101}/p_{001}\ p_{100}) \\
u_{XZ} &= log(p_{000}\ p_{011}/p_{001}\ p_{010}) \\
u_{YXZ} &= log(p_{001}\ p_{010}\ p_{100}\ p_{111}/\ p_{000}\ p_{011}\ p_{101}\ p_{110})
\end{aligned}
$$

# Appendix

Parameter values for the figures.

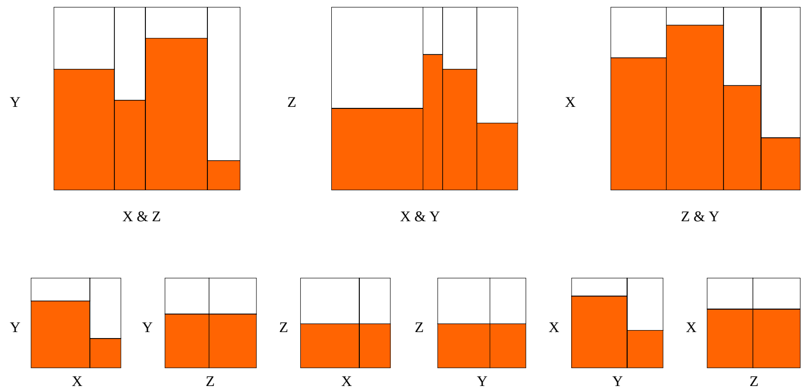



Figure 10: No flats. Two marginal independences. No other independence

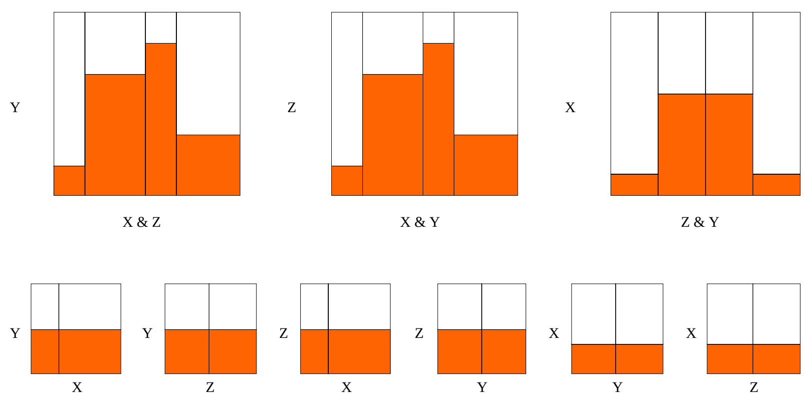



Figure 11: No flats. Three marginal independences. No other independence

| Figure | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| 3(a) | 2/7 | 8/35 | 1/7 | 12/35 | 4/5 | 1/2 | 3/10 | 7/10 |
| 3(b) | 2/7 | 8/35 | 1/7 | 12/35 | 4/5 | 1/2 | 3/5 | 1/5 |
| 3(c) | 2/7 | 8/35 | 1/7 | 12/35 | 4/5 | 1/2 | 4/5 | 1/2 |
| 4 | 2/7 | 8/35 | 1/7 | 12/35 | 7/10 | 7/10 | 7/10 | 7/10 |
| 5 | 10/33 | 4/11 | 5/33 | 2/11 | 7/10 | 7/10 | 7/10 | 7/10 |
| 6 | 1/3 | 1/6 | 1/3 | 1/6 | 7/10 | 7/10 | 3/10 | 3/10 |
| 7 | 2/9 | 1/9 | 2/9 | 4/9 | 2/3 | 2/3 | 1/6 | 1/6 |
| 8 | 1/7 | 1/7 | 3/7 | 2/7 | 1/3 | 2/3 | 1/4 | 1/6 |
| 9 | 1/4 | 1/4 | 1/4 | 1/4 | 3/4 | 1/2 | 1/4 | 1/4 |
| 10 | 1/3 | 1/6 | 1/3 | 1/6 | 2/3 | 1/2 | 5/6 | 1/6 |
| 11 | 1/6 | 1/3 | 1/6 | 1/3 | 1/6 | 2/3 | 5/6 | 1/3 |

## References

Cherry, W.H. and R.W. Oldford (2002). "On the poverty of Venn diagrams for teaching probability: their history and replacement by Eikosograms." *(submitted for publication)*
Cox, D.R. and N. Wermuth (1996). *Multivariate Dependencies*. Chapman and Hall, London.
Lauritzen, S. (1996). *Graphical Models*. Oxford University Press, Oxford.
Whittaker, J. (1990). *Graphical models in applied multivatiate statistics*. John Wiley & Sons, Chichester.