Results for option pricing

- \([o,v,b]=\text{optimal}(\text{rand}(1,100000))\)
- Estimators = 0.4619  0.4617  0.4618  0.4613  0.4619
- \(o = 0.46151\) % best linear combination (true value=0.46150)
- \(v = 1.1183e-005\) %variance per uniform input
- \(b' = -0.5503 \ 1.4487 \ 0.1000 \ 0.0491 \ -0.0475\)
Efficiency of Optimal Linear Combination

• Efficiency gain based on number of uniform random numbers $0.4467/0.00001118$ or about 40,000.
• However, one uniform generates 5 estimators requiring 10 function evaluations.
• Efficiency based on function evaluations approx 4,000

• A simulation using 500,000 uniform random numbers; 13 seconds on Pentium IV (2.4 Ghz) equivalent to twenty billion simulations by crude Monte Carlo.
Interpreting the coefficients $b$. Dropping estimators.

• Variance of the mean of 100,000 is $1.18 \times 10^{-10}$
  Standard error is around .00001
• Some weights are negative, (e.g. on $Y_1$) some more than 1 (on $Y_2$), some approximately 0 (could they be dropped? For example if we drop $Y_3$ then

  variance increases to about $1.6 \times 10^{-10}$. 
Tips…

• If you are simulating to generate results for a more complicated model (e.g. asian option, non-normal distribution etc) use a simple model (European option, normal distribution etc) as control variate. Use simulation to estimate the difference (assuming you know the result for the control)

• Allow the uniform variates as input parameters. This facilitates variance reduction without changing the program.

• Try a variety of variance reduction techniques (5-10) including some with second-difference like expressions. Your best estimator is usually an optimal linear combination.

• Only combine antithetic random numbers additively.
Black-Scholes price in R

- \text{blsprice} = \text{function}(\text{So}, \text{strike}, r, T, \text{sigma}, \text{div})\
  \text{d1} <- (\log(\text{So}/\text{strike}) + (r - \text{div} + (\text{sigma}^2)/2) * T)/(\text{sigma} * \sqrt{T})\
  \text{d2} <- \text{d1} - \text{sigma} * \sqrt{T}\
  \text{call} <- \text{So} * \exp(-\text{div} * T) * \text{pnorm} (\text{d1}) - \exp(-r * T) * \text{strike} * \text{pnorm} (\text{d2})\
  \text{put} = \text{call} - \text{So} + \text{strike} * \exp(-r * T)\
  \text{c} (\text{call}, \text{put})
Useful URL’s

• http://www.std.com/nr/index.html (numerical recipies)
• http://www.cboe.com
• http://rweb.stat.umn.edu/R/ (R library)
Simulating Survivorship bias and the maxima of Brownian Motion

Examples in Biostatistics: Sequential tests for a mean. e.g. For testing hypothesis

\[ H_0 : \mu \geq 0 \]

\[ H_1 : \mu < 0 \]

Reject \( H_0 \) as soon as \( B(t) < -c_0 - c_1 t \).

i.e. \( \min \{B(t) + c_0 + c_1 t < 0\} \)
Modeling highs

Our model shows that stocks will go higher, just not which ones and when.
Brownian Motion

- Brownian motion

\[ dX(t) = \mu dt + \sigma dW(t) \]

\[ \Delta X(t) \sim N(\mu \Delta t, \sigma^2 \Delta t) \]
We typically observe

\[ O_i = X_i(0) \]
\[ C_i = X_i(T) \]
\[ L_i = \min\{ X_i(t); 0 < t < T \} \]
\[ H_i = \max\{ X_i(t); 0 < t < T \} \]

\( X_i(t) \) are correlated Brownian Motion processes
Acceptance-rejection generating from $f(x)$

Suppose we generate the Close C by Acceptance-Rejection using its pdf $f(x)$
We can use the same picture to simulate jointly \((H, C)\) for BM.
Similarly if C is discrete
Exponential Statistics

Theorem:
For Brownian Motion,
\[ Z_H = (H - O)(H - C) \sim \exp\left(\frac{\sigma^2 T}{2}\right) \]
\[ Z_L = (L - O)(L - C) \sim \exp\left(\frac{\sigma^2 T}{2}\right) \]
independent of (O, C).

Proof:
Random Walk and reflection

\[ P(H \geq m \mid C = u) = \begin{cases} 
1 & m \leq \max(0, u) \\
\frac{f(2m-u)}{f(u)} & m > u
\end{cases} \]
Substitute observed values in survivor function in Normal case

\[ U = \frac{f(2H - C)}{f(C)} \] is the survivor function for \( H \) given \( C \) evaluated at its observed values. Therefore it is Uniform[0,1].

But \( U = \frac{f(2H - C)}{f(C)} = \exp\left\{ \frac{1}{2T \sigma^2} (C^2 - (2H - C)^2) \right\} \)

\[ = \exp\left\{ -\frac{2H(H - C)}{T \sigma^2} \right\} \]

Therefore \(- \ln(U) = \frac{2H(H - C)}{T \sigma^2}\) is \( \exp(1) \)

\[ Z_H = H(H - C) \text{ is } \exp\left( \frac{T \sigma^2}{2} \right). \]
Estimating volatility based on High and Low

Both $Z_H$ and $Z_L$ (and average) provide an estimator of volatility or variance. Alternative to sample variance of increments e.g. average$(C - O)^2$

average$(Z_H + Z_L)$ has about five times the efficiency

One week high and low equivalent to daily data. Also unbiased regardless of drift in BM.
Cumulative Variance

2001: US/CAD Exchange rate
US/CAD$ Exchange rate
Volatility 1995-2001

High vol begins Aug 27 '98
Evident from graph of exchange rate?
Moving average Volatility

Russian debt crisis
Are ZH and ZL exponential distributed? US/CAD\$ exchange & Dow Jones

To transform from Geometric to Brownian Motion,

\[
Z_H = \ln\left( \frac{H}{O} \right) \left( \frac{H}{C} \right)
\]

\[
Z_L = \ln\left( \frac{L}{O} \right) \ln\left( \frac{L}{C} \right)
\]
In practice, measures of volatility may differ. e.g. DJA

- Intra-day volatility (High-Low estimator) >> inter-day vol (open-close)
Risk Manager

"Let's see... bungee jumping at noon, sky diving 1:30, alligator wrestling at 3:00. To hedge I will short Air Canada and cancel dinner with my mother-in-law."
SURVIVORSHIP BIAS

- E.g. Retrospective studies on performance of mutual fund managers. Present market value is conditional on low > 0.
What happens to the mean of the survivors as the variance increases?

- survivemovie