CONTROL VARIATES

- Suppose there is another function *g(u)* that resembles *f(u)* but simpler for which I can compute the integral. Use it to improve the Monte Carlo integration of *f.*

$$\int_0^1 f(u)du = \int_0^1 g(u)du + \int_0^1 [f(u) - g(u)]du$$

Use the known value for the first term $\int_0^1 g(u)du$

and estimate the second term, the difference, using Monte Carlo.

Estimator is

$$\int_0^1 g(u)du + \frac{1}{n}\sum_{i=1}^{n}(f(U_i) - g(U_i))$$

Control variates

Estimator is

$$\int_0^1 g(u)du + \frac{1}{n}\sum_{i=1}^{n}(f(U_i) - g(U_i))$$

VARIANCE

$$\frac{1}{\mathrm{n}}\mathrm{var}(f(U_1) - g(U_1))$$

This is better than crude if $\mathrm{var}(f(U_1) - g(U_1)) < \mathrm{var}(f(U_1))$

When $g(U_1)$ is close to $f(U_1)$ variance of difference is small.

Control Variate

I tried a function of the form

$$g(u) = (u - .47) + 6(u - .47)^2 \text{ if } u > .47$$

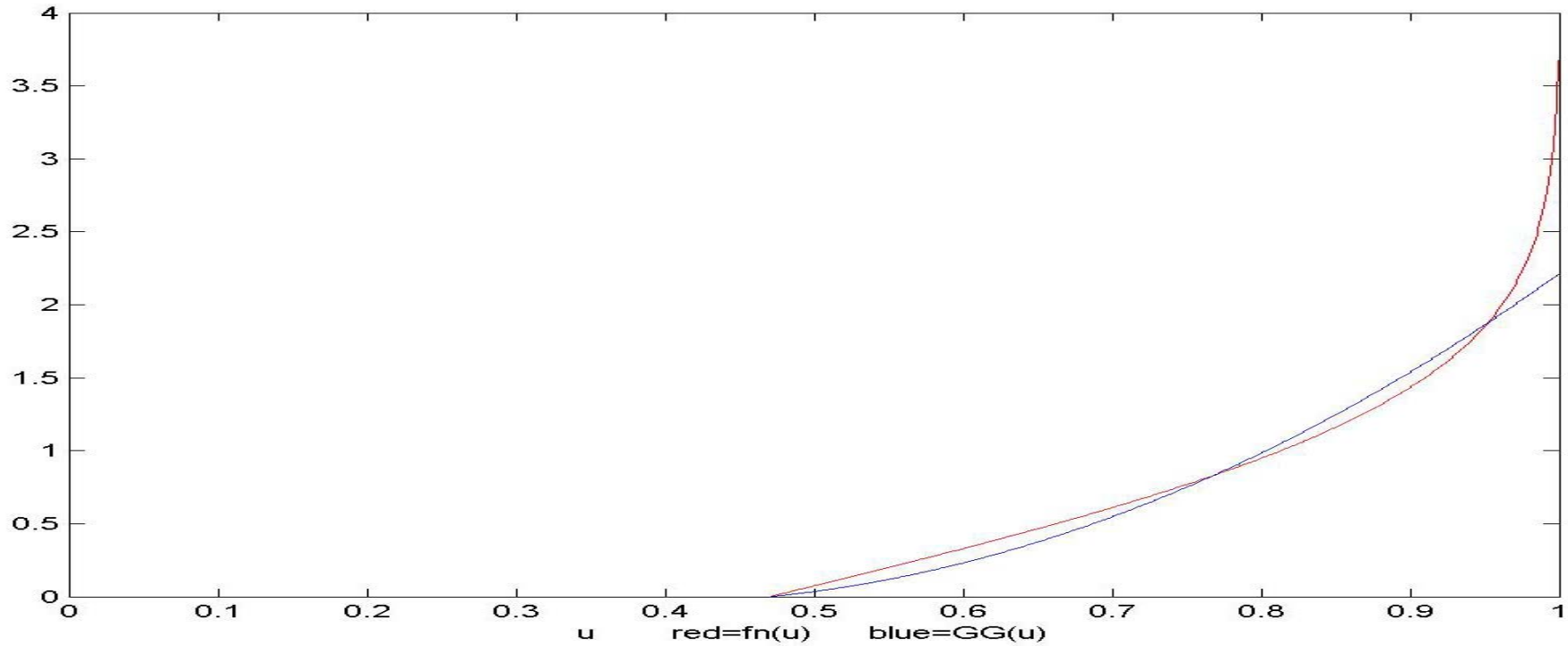
$$g(u) = 0 \text{ if } u \leq .47.$$

This is easy to integrate since it is quadratic.

- function g=GG(u)

```
% control variate for callopt2.
% this function integrates to 2*(.53)^3+.53^2/2
u=max(0,u-.47);
g=6*u.^2+u;
```

Function GG(u) and fn(u)



- Only need to estimate the area between the curves by Monte Carlo

Performance of Control Variate

- u= rand(1,500000);
- F=fn(u);
- G=GG(u);
- est=2*(.53)^3+.53^2/2+(F-G);
- mean(est)
- var(est)/length(est)
- (mean= 0.4616 var= 2.93e-008)
- Efficiency compared with crude 8.7e-007 ratio=30 (equivalent to 15 million crude!)

Control Variates II

The "best" approximation to f using g is not necessarily g itself.

As an alternative, we can approximate *f(U)* using a linear regression on *g(U)*

$f(U) = \theta + \beta(g(U) - Eg(U)) + \varepsilon,$ ε is a mean zero "error" random variable.

where $\theta = E(f(U)) = \int_0^1 f(u)du$

Slope parameter given by $\beta = \dfrac{\mathrm{cov}(f(U), g(U))}{\mathrm{var}(g(U))}$.

We wish to estimate the parameter θ and since

$\theta = E[f(U) - \beta(g(U) - Eg(U))]$ we can use a sample mean of

terms of this form, i.e.

$$\frac{1}{n}\sum_{i=1}^{n}[f(U_i) - \beta(g(U_i) - Eg(U_i))].$$

$$= \beta E(g(U)) + \frac{1}{n}\sum_{i=1}^{n}[f(U_i) - \beta(g(U_i))].$$

This is identical to the earlier Control Variate if $\beta = 1$.

However since β is usually unknown, we replace it by the regression estimator of slope, equivalent to replacing variance and covariance by sample variance and covariance .

$$\hat{\beta} = \frac{\sum_{i=1}^{n} [f(U_i) - \overline{f(U)}][g(U_i) - \overline{g(U)}]}{\sum_{i=1}^{n} [g(U_i) - \overline{g(U)}]^2} \quad \text{where } \overline{g(U)} = \frac{1}{n}\sum_{i=1}^{n} g(U_i).$$

It is easy to show that the variance of this control estimator

$$\mathrm{var}(\beta E(g(U)) + \frac{1}{n}\sum_{i=1}^{n}[f(U_i) - \beta(g(U_i))]) = (1 - r^2)\,\mathrm{var}(\frac{1}{n}\sum_{i=1}^{n} f(U_i))$$

so the efficiency is given by $\frac{1}{1 - r^2}$ where

r = correlation coefficient between $f(U_i)$ and $g(U_i)$.

Example of Control Variate II

Find $\int_0^{10} \frac{x}{(2+x^{1/4})} dx = E(f(X))$ where X is U[0,10] and

$f(x) = \frac{10x}{(2+x^{1/4})}$. consider as control variate $g(x) = x$.

Note that $E(g(X)) = E(10U) = 5$.

- x=10*rand(1,500000);
- f=10*x./(2+x.^.25); g=x; C=cov([f' g']);
- beta=C(1,2)/C(2,2); EG=5;
- est=beta*EG+f-beta*g;
- mean(est); var(est)/length(est)
- (mean= 14.000011 var= 1.8e-007)
- var(f)/500000= 2.8160e-004
- Eff=622 (equivalent to 311 million crude)

Stratified Sample

- Consider choosing one point in interval [0,a] and another in the interval [a,1]

aU_1 is in [0, a] and $a+(1-a)U_2$ is in [a,1].

Consider using a weighted average of the function at these two points

to estimate $\int_0^1 f(u)du$. What should the weights be?

Let $\theta_{ST} = af(aU_1)+(1-a)f(a+(1-a)U_2)$

where U_1, U_1 are independent U[0,1]. Why weights $a, 1-a$?

$$E(\theta_{ST}) = E(af(aU_1)+(1-a)f(a+(1-a)U_2)) = aE(f(aU_1)+(1-a)Ef(a+(1-a)U_2$$

$$= a\int_0^1 f(au)du + (1-a)\int_0^1 f(a+(1-a)u)du = \int_0^a f(u)du + \int_a^1 f(u)du = \int_0^1 f(u)du$$

Weights in a stratified sample

- The weights attached to a given stratum (interval, region) are proportional to the length of the interval (or volume of region).
- We may allow many observations in a given stratum and apply weight to the stratum mean.

General stratified sample

- Choose

n_i from stratum $i, i = 1,2,...m$

Evaluate the function at n_i random points, uniformly distributed over stratum i and average the result

$$AV_i = \frac{1}{n_i}\sum_{j=1}^{n_i} f(V_{ij}) \text{ where } V_{ij} \text{ is uniformly distributed over stratum i.}$$

Use WEIGHTED AVERAGE of the stratum mean, weight proportional to length, (area, or volume) of stratum.

Variance of Stratified Sample

Estimator:

$$\sum_i (x_{i+1} - x_i) AV_i$$

Variance of Stratified Sample estimator:

$$\sum_i (x_{i+1} - x_i)^2 \frac{\mathrm{var}(f(V_{ij}))}{n_i}$$

Performance of stratified sample

- Example. We chose a=.7 and sampled 100,000 on [0,0.7] and another 100,000 on [0.7,1].
- a=.7;
- F=a*fn(a*rand(1,500000))+(1-a)*fn(a+(1-a)*rand(1,500000));
- mean(F) % =0.4608)
- var(F)/length(F) % =9.18e-008
- Compare with Crude with n=1000000. Variance= 4.3e-007 ; Efficiency around 5.

Optimal sample sizes for stratified sample

The optimal sample size in each stratum is proportional to

(a) the size of the stratum (e.g. interval length)

(b) the stratum standard deviation

So if stratum i is the interval (x_i, x_{i+1}) then

optimal sample size n_i proportional to

$$(x_{i+1} - x_i)\sqrt{\mathrm{var}(f(x_i + U(x_{i+1} - x_i))}$$

$\mathrm{var}(f(x_i + U(x_{i+1} - x_i))$ estimated from a preliminary sample.

The function “stratified”

```
function [est,v,n]=stratified(x,nsample)
% input x=vector of strata (e.g. x=[0 .25 .5 .75 1] and nsample=sample size
est=0;    n=[];         m=length(x);
for i=1:m-1    %preliminary simulation of 10000 to optimize sample size
   v= var(fn(unifrnd(x(i),x(i+1),1,1000));
   n=[n (x(i+1)-x(i))*sqrt(v)];
end
n=floor(nsample*n/sum(n));    % these are stratum sample sizes
v=0;
for i=1:m-1
   F=fn(unifrnd(x(i),x(i+1),1,n(i)));
   est=est+(x(i+1)-x(i))*mean(F);
   v=v+var(F)*(x(i+1)-x(i))^2/n(i);
   end
```

Stratified Sample with more strata, optimal sample size

- Try 4 strata [0 .55] ,[.55 .8],[.8 .94],[.94,1] with optimal sample sizes
  - 32717 78703 46420 42158
- [est,v,n]=stratified([0 .55 .80 .94 1],500000)
- estimator 0.4617 variance= 3.7e-008
- Efficiency now around 24 times that of Crude. Try also:
- [est,v,n]=stratified([.47 .62 .75 .87 .96 1],500000)

(var=1.4e-8)

Conditioning as a Variance Reduction Tool

- Example: suppose we wish to find the area of a region, say the area under the graph of a function *h(x).*
- Crude Monte Carlo: Find a probability density function *g(x)* such that

$$cg(x) \geq h(x) \text{ for all } x. \text{ Then generate random points}$$

(X_i, Y_i) uniformly distributed under the graph of $cg(x)$:

X_i has p.d.f. $g(x)$ and $Y_i = cg(X_i)U_i$.

The proportion of these points which are also under the graph of $h(x)$

$$\rightarrow \frac{\text{area under } h(x)}{\text{area under } cg(x)} = \frac{\text{area under } h(x)}{c}$$

Conditioning (II)

- Leads to the estimator:

$$\hat{\theta}_{CR} = \frac{\text{c} \times \text{Number points under h}}{\text{Total number of points}} = \frac{c}{n}\sum_{i=1}^{n} I(Y_i \le h(X_i)) = \frac{c}{n}\sum_{i=1}^{n} Z_i, \text{ say}$$

Consider an alternative estimator

$$\hat{\theta}_{Co} = \frac{c}{n}\sum_{i=1}^{n} E[Z_i \mid X_i] = \frac{c}{n}\sum_{i=1}^{n} \frac{h(X_i)}{cg(X_i)} = \frac{1}{n}\sum_{i=1}^{n} \frac{h(X_i)}{g(X_i)}$$

We obtained this new estimator by taking the conditional expectation of the old.

Does this always give an unbiased estimator with smaller variance?

Some Properties of Conditional Expectation

1. $E\{E[Y \mid X]\} = E(Y)$

2. $\mathrm{Var}(\mathrm{E}[\mathrm{Y} \mid \mathrm{X}]) \leq \mathrm{var}(\mathrm{Y})$ with equality if and only if Y is a function of X.

IF $\hat{\theta}$ is an unbiased estimator and for some random variable X we can compute $E(\hat{\theta} \mid \mathrm{X})$ then this is another unbiased estimator, typically with smaller variance. Therefore

$\mathrm{var}(\mathrm{Z_i}) \geq \mathrm{var}(E(Z_i \mid X_i))$ and the second estimator

$\frac{c}{n}\sum_{i=1}^{n} E[Z_i \mid X_i]$ is better than the first $\frac{c}{n}\sum_{i=1}^{n} Z_i$

Example of Conditioning: Estimating pi

If we generate (U, V) uniform on $[0,1]$, and if

$Z = I(U^2 + V^2 \le 1)$ then $E(Z) = \frac{\pi}{4}$ and

$E(Z \mid V) = (1 - V^2)^{1/2}$. Compare the crude estimator

$\frac{4}{n}\sum_{i=1}^{n} Z_i$ with the conditional expectation $\frac{4}{n}\sum_{i=1}^{n} E(Z_i \mid V_i)$

```
u=rand(2,200000);  z= 4*(sum(u.^2)<1); theta1=mean(z);
var(z)/length(z)

z2=4*sqrt(1-u(2,:).^2); theta2=mean(z2);  var(z2)/length(z2)
```

Efficiency gain is about 3.3

Combining conditioning and stratified sample

We estimated

$\pi = 4\int_0^1 (1-u^2)^{1/2}\,du$ using the conditional expectation estimator

$\frac{4}{n}\sum_{i=1}^{n}(1-V_i^2)^{1/2}$. What if we use a stratified sample with one observation in each interval

$(\frac{\text{i -1}}{\text{n}}, \frac{i}{n}), i = 1,2,\ldots,n.$ Result is $\frac{4}{n}\sum_{i=1}^{n}(1-(\frac{i-1+V_i}{n})^2)^{1/2}$

```
n=500000; u=rand(1,n);   w=((0:(n-1))+u)/n;

z3= 4*sqrt(1-w.^2);  theta3=mean(z3) ; pi-theta3
```

Accurate to 8 decimals!

IMPORTANCE SAMPLING

- Idea: Why not weight observations in different parts of the sample space differently?
- e.g. interested in behaviour of a communication system or network under heavier than usual loads. Generate simulation under heavy loads and adjust the results for the difference.

Importance Sampling

- Can I increase the probability of the samples that most effect my estimator without creating bias?
- Use importance sampling when:
  - we want to find $\int f(u)du$
  - We can find a probability density function $g(u)$ such that it is easy to generate random variables with this probability density function and *g* is **roughly proportional to *f*** so that *f(u)/g(u)* is reasonably close to constant.

Importance II

- to find an appropriate importance sampling distribution, start with a function *g* similar to *f* that is non-negative and easily integrated. Then multiply this function by a constant to obtain a p.d.f. Then:

$$\theta = \int f(u)du = \int \frac{f(z)}{g(z)} g(z)dz = E[\frac{f(Z)}{g(Z)}]$$

Where Z is a random variable having probability density function $g(z)$.

We should therefore estimate θ using

$$\theta_{\mathrm{IM}} = \frac{1}{n}\sum_{i=1}^{n} \frac{f(Z_i)}{g(Z_i)}$$ where the random variable Z_i are generated with probability density function $g(z)$. For example $Z_i = G^{-1}(U_i)$ where G is the c.d.f. of Z.

Variance of Importance sample Estimator

$$\theta = \int f(u)du = \int \frac{f(z)}{g(z)} g(z)dz$$

$$\theta_{\mathrm{IM}} = \frac{1}{n}\sum_{i=1}^{n} \frac{f(Z_i)}{g(Z_i)} \text{ where } Z_i \text{ independent with p.d.f. } g(z).$$

$$Var(\theta_{\mathrm{IM}}) = \frac{1}{n}\mathrm{var}(\frac{f(Z_i)}{g(Z_i)})$$

More generally to estimate $E[h(X)]$ where X has p.d.f. $f(x)$, use

$$E[h(X)] = E[h(Z)\frac{f(Z\ )}{g(Z\ )}] \text{ where } Z \text{ has p.d.f. } g(z). \text{ The estimator is}$$

$$\theta_{\mathrm{IM}} = \frac{1}{n}\sum_{i=1}^{n} h(Z_i)\frac{f(Z_i)}{g(Z_i)} \text{ with } Z_i \text{ independent, p.d.f. } g(z).$$

$$Var(\theta_{\mathrm{IM}}) = \frac{1}{n}\mathrm{var}[h(Z_i)\frac{f(Z_i)}{g(Z_i)}]$$

Example: importance sampling

- Find $\int_0^1 x^{1.6} e^{-x} dx.$ The true value is around 0.19113.
- Crude: u=rand(1,200000);
  - mean(u.^1.6.*exp(-u)) % =0.1916
    var(u.^1.6.*exp(-u))/200000 %=6.7676e-008
- Importance:

$$g(x) = 2.6x^{1.6} \text{ for } 0 < x < 1. \quad G(x) = x^{2.6} \text{ and } G^{-1}(U) = U^{1/2.6}$$

$$\theta_{IM} = \frac{1}{n}\sum_{i=1}^{n} \frac{f(Z_i)}{g(Z_i)} = \frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2.6} e^{-Z_i} \text{ where } Z_i = U_i^{1/2.6}.$$

Importance sampling example

- u=rand(1,200000);
- theta_IM=exp(-u.^(1/2.6))./2.6;
- mean(theta_IM)
- var(theta_IM)/200000 %= 9.3300e-009
- (about 7 times as efficient as crude)

Importance Sampling in Option Pricing example

- For the call option, try and approximate f with a linear function starting at (.47,0).

$g(x) = k(x - .47)^+$ where we choose k so this is a p.d.f.

c.d.f. $G(x) = \frac{k}{2}[(x - .47)^+]^2$ so $k = 7.12$

$G^{-1}(U) = .47 + .53\sqrt{U}$ generates Z from the importance sampling density.

Why not use a density function (e.g. a quadratic density) that more closely approximates the function *f*?

Matlab Code for Importance Sample. Stock Option Example.

- function [est,v]=importance(f,g,Ginv,u)
- %runs a simulation on the function 'f" using importance density "g"(both character strings) and inverse c.d.f. "Ginverse“
- e.g. [est,v]=importance('fn','importancedens','Ginverse',rand(1,200000));
- % outputs all estimators (use mean(est)) and variance of estimator.
- % IM is the inverse cf of the importance distribution c.d.f.
- IM= eval(strcat(Ginv,'(u)')); %=.47+.53*sqrt(U);
- %IMdens is the density of the importance sampling distribution at IM
- IMdens=eval(strcat(g,'(IM)')); %2*(IM-.47)/(.53)^2;
- FN=eval(strcat(f,'(IM)'));
- est=FN./IMdens; % mean(est) provides the estimator
- v=var(FN./IMdens)/length(IM); % this is the variance of the estimator
- **********
- %GIVES mean(est)= 0.4614 v = 6.4329e-008
- EFFICIENCY GAIN OVER CRUDE IS AROUND 35.

Exponential tilting and importance sampling: Estimating the probability of rare events

Suppose we wish to estimate an extreme quantile such as VAR 99.9% . Then we need to estimate the probability that a random variable exceeds some large value. i.e. when

$S_{10} > a$ where $S_{10} = \sum_{i=1}^{10} X_i$ for independent random variables

X_i all have probability density function $f(x)$. This is

$E\{I(S_{10} > a)\}$ and is estimate by the corresponding sample mean.

If we conduct the simulation

under this distribution very few of the sums will exceed a. What if we generate X_i

as independent random variables under a different (tilted) probability density function

$g(x) = \frac{e^{tx}}{k} f(x)$ where k is chosen so that this is a density function and then use

importance sampling.

For example suppose that $f(x) = e^{-x}$. Then the tilted distribution is

$g_t(x) = \frac{1}{k} e^{tx-x} = e^{-x(1-t)} \times (1-t)$ since this is the multiple

that makes this function a p.d.f.... another exponential distribution.

If we generate X_i from this p.d.f. it has

expected value $\theta = \frac{1}{1-t}$. The importance sample estimator is

$$E\{I(\sum_{i=1}^{n} Z_i > a)\prod_{i=1}^{n} \frac{f(Z_i)}{g_t(Z_i)}\} = E\{I(\sum_{i=1}^{n} Z_i > a)\prod_{i=1}^{n} \theta e^{-tZ_i}\} = \theta^n E\{I(\sum_{i=1}^{n} Z_i > a) e^{-t\sum_{i=1}^{n} Z_i}\}$$

where Z_i are independent with exponential(θ) p.d.f.

What is the best value of θ? The variance of the importance sample estimator is

small when we choose t so that $\sum_{i=1}^{n} Z_i$ is approximately a. Choose t = 1-1/θ so that

$E(Z_i) = \theta = a/n$ so choose θ approximately a/n.

Matlab Code for Example, a=22.6, n=10

- N=500000
- X=exprnd(1,10,N);
- est1=(sum(X)>22.6);
- mean(est1) % 0.001
- var(est1) % 8.8e-004

Using Importance sampling

- Z=exprnd(2.5,10,N);
- S=sum(Z); theta=2.5
- est2=theta^10*exp(-(1-1/theta)*S).*(S>22.6);
- mean(est2) % =0.001
- var(est2) % 5.5e-6, and SE about 0.000003
- ***EFFICIENCY GAIN FROM IMPORTANCE SAMPLING IS ABOUT 180.***

Common Random Numbers: estimating a difference

- Suppose we wish to estimate the difference between two systems or values: e.g.
  - Estimate the slope of a function at a point
  - Estimate the difference in performance between two pieces of equipment
  - Estimate the difference between a call option price for r=0.05 and r=0.06.

Importance Sampling, Tilting and the Saddlepoint Approximation

The Edgeworth Expansion of the probability density function of the sample mean of i.i.d. random variables takes the form

$$n(x;\mu,\sigma^2)\{1+\frac{\kappa}{6\sqrt{n}}[z^3-3z]+O(1/n)\}$$

where $z=\sqrt{n}(x-\mu)/\sigma$ and $\kappa=E(X-\mu)^3$, and

$n(x;\mu,\sigma^2)$ is the usual normal density function approximation.

Notice that at $x=\mu$, the correction is zero and the normal approximation is accurate to O(1/n).

Consider tilting the original distribution so as to achieve this...

If we wish to approximate the proability density function $f(x)$ of X at the point x and X has cumulant generating function $K(t)=\ln E\{\exp(Xt)\}$, then

Saddlepoint(cont)

the density $f_\theta(x) = e^{\theta x - K(\theta)} f(x)$ has mean given by $K'(\theta)$ so if we choose θ so that $K'(\theta) = x$ the mean is x and the variance $K''(\theta)$.

The Edgeworth approximation to the density f_θ is $n(K'(\theta), K'(\theta), K''(\theta)) = \dfrac{1}{\sqrt{2\pi K''(\theta)}}$

Therefore the (saddlepoint) approximation to $f(x)$ is

$$\frac{1}{\sqrt{2\pi K''(\theta)}} e^{K(\theta) - \theta x}, \text{ where } K'(\theta) = x.$$

Two steps of saddlepoint

- Saddlepoint approximation has two steps
  - Tilt the original density so that mean of tilted distribution is around the point of interest x
  - Use the normal approximation to the tilted distribution.
- Importance sampling from the tilted density avoids the second step.

Tilting and large deviations (Robert and Casella p 132)

Large deviations results concern the tail of a distribution of partial sums. e.g.

$P[S_n > x]$ when this probability is very small.

Cramer's Theorem.

Put $I(x) = \theta x - K(\theta)$ where again θ is such that $K'(\theta) = x$. Then

$$\frac{1}{n}\ln(P[S_n > x]) \sim -I(x)$$

Student presentation 11. Importance Sampling (Robert and Casella 85-87)

- Suppose X as the student t distribution with 12 degrees of freedom. Estimate

$$E\left(\frac{X^5}{1+(X-3)^2} I(X \geq 0)\right)$$

using crude Monte Carlo and imporance sampling and various choices of the importance distribution including Cauchy, U[0,1/2.1], and normal distribution. Which do you prefer and why?

Reducing Variance when estimating the difference of Expected values

In general we wish to estimate

$E(h_1(X)) - E(h_2(Y))$

where X, Y have cumulative distributi on functions

F_X and F_Y respective ly. Then

$\mathrm{var}[h_1(X) - h_2(Y)] = \mathrm{var}[h_1(X)] + \mathrm{var}[h_2(Y)] - 2\,\mathrm{cov}(h_1(X), h_2(Y))$

and this is small if $\mathrm{cov}(h_1(X), h_2(Y))$ is large.

When functions $h_1(X), h_2(Y)$ are both increasein g functions

we can arrange a large covariance if we use the

SAME uniform to generate X and Y *.i.e.*

$X = F_X^{-1}(U),\ Y = F_Y^{-1}(U).$

Opposite of antithetic .

Theorem: Common and Antithetic Random Numbers

Theorem : Suppose $h_1(x), h_2(y)$ are both increasing or both decreasing functions. Subject to the constraints that X, Y have given cumulative distribution functions F_X and F_Y respectively, then the covariance $\mathrm{cov}(h_1(X), h_2(Y))$ is *maximized* when we use common random numbers, i.e.

$$X = F_X^{-1}(U),\ Y = F_Y^{-1}(U)$$

and *minimized* when we use antithetic random numbers :

$$X = F_X^{-1}(U),\ Y = F_Y^{-1}(1-U)$$

Comparing estimators for Call Option Pricing Example

- script9

Combining Monte Carlo Estimators

- If I have many MC estimators, with/without various variance reduction techniques, which should I choose?

Combining Estimators

- Suppose I have *m* unbiased estimators all of the same parameter θ
- Put these estimators in a vector *Y*

$Y = (Y_1, Y_2, ... Y_m)'$ so that $\mathrm{E}(Y) = 1\theta$ where 1 represents the vector $(1,1,...,1)$ of length m.

Any linear combination of these estimators with coefficients that add to one is also an unbiased estimator of the parameter θ

Which such linear combination is best?

Best linear combination of estimators.

Q: What linear combination of these estimators does the best job of estimating θ.

A: The best linear combination is $\sum_i b_i Y_i$ where $\sum b_i = 1$ and

b_i is proportional to $\sum_{j=1}^{m} C_{ij}$ and where $C = V^{-1}$ is the inverse of the covariance matrix V of Y.

$V_{ij} = \mathrm{cov}(Y_i, Y_j)$.

Estimating Covariance

When we have many independent values of these estimators, (from n simulations, e.g. Y_{ij}, $j = 1,2,\ldots,n$ are replicated values of $Y_{i,}$

we may estimate variance-covariance matrix using sample covariance:

Estimate V_{ij} by

$$\frac{1}{n-1}\sum_{k=1}^{n}(Y_{ik} - \bar{Y}_i)(Y_{jk} - \bar{Y}_j)$$

Theorem on Optimal Linear Combination of estimators

$Theorem$: (Best Linear Combination of Estimators)

The linear combination of estimators Y_i, $\mathrm{i} = 1,2,...\mathrm{m}$ of the form

$\sum_{i=1}^{m} b_i Y_i$ where the vector b is given by

$b' = (1'V^{-1}1)^{-1}1'V^{-1}$. Here 1 is the column vector of m ones.

The variance of the resulting estimator is

$$b'V^{-1}b = \frac{1}{1'V^{-1}1}.$$

Example: combining the estimators of the call option price

Consider the following estimators:

$$Y_1 = \frac{0.53}{2}(f(.47+.53U)+f(1-.53U)) \text{ (antithetic)}$$

$$Y_2 = \frac{0.37}{2}[f(.47+.37U)+f(.84-.37U)]$$

$$+\frac{.16}{2}[f(.84+.16U)+f(1-.16U)]$$

This is stratified into $[.47,.84]$ and $[.84,1]$ and uses antithetic within strata, common U between strata.

$$Y_3 = .37 f(.47+.37U)+.16 f(1-.16U)$$

stratified, antithetic between strata

Example: (cont)

$$Y_4 = \int g(u)du + [f(U) - g(U)] \text{ (control variate)}$$

where $g(u) = 6[(u - .47)^+]^2 + (u - .47)^+$

$$Y_5 = \frac{f(Z)}{g(Z)} \text{ where } Z = .47 + .53\sqrt{U} \text{ (importance sampling)}$$

Generate simulated values of all five estimators $Y_1, \ldots Y_5$ using the same uniform.

Do this repeatedly for n values of U. Obtain the individual estimators and the covariance matrix V. The best linear combination is $\sum_{i=1}^{5} b_i \overline{Y_i}$

where the vector b is given by $b = \frac{1}{1'V^{-1}1} V^{-1}1$. ($b$ is proportional to the sum of the rows of V^{-1} - rescaled so $\sum b_i = 1$)

MATLAB function *OPTIMAL*

```
function [o,v,b,t1]=optimal(U)
% generates optimal linear combination of five estimators and outputs
% average estimator and variance.
t1=cputime;
Y1=(.53/2)*(fn(.47+.53*U)+fn(1-.53*U));          t1=[t1 cputime];
Y2=.37*.5*(fn(.47+.37*U)+fn(.84-.37*U))+.16*.5*(fn(.84+.16*U)+fn(1-
.16*U));
t1=[t1 cputime];
Y3=.37*fn(.47+.37*U)+.16*fn(1-.16*U);          t1=[t1 cputime];
intg=2*(.53)^3+.53^2/2;          Y4=intg+fn(U)-GG(U);          t1=[t1
cputime];
Y5=importance('fn','importancedens','Ginverse',U);          t1=[t1
cputime];
X=[Y1' Y2' Y3' Y4' Y5'];
mean(X)
V=cov(X);      Z=ones(5,1);      C=inv(V);      b=C*Z/(Z'*C*Z);
o=mean(X*b);      % this is mean of the optimal linear combinations
t1=[t1 cputime];
v=1/(Z'*V1*Z);
t1=diff(t1);   % these are the cputimes of the various estimators.
```

Results for option pricing

- [o,v,b]=optimal(rand(1,100000))
- Estimators = 0.4619 0.4617 0.4618 0.4613 0.4619
- o = 0.46151 % best linear combination (true value=0.46150)
- v = 1.1183e-005 %variance per uniform input
- b’ = -0.5503 1.4487 0.1000 0.0491 -0.0475

Efficiency of Optimal Linear Combination

- Efficiency gain based on number of uniform random numbers 0.4467/0.00001118 or about 40,000.
- However, one uniform generates 5 estimators requiring 10 function evaluations.
- Efficiency based on function evaluations approx 4,000
- *A simulation using 500,000 uniform random numbers ; 13 seconds on Pentium IV (2.4 Ghz) equivalent to* ***twenty billion*** *simulations by crude Monte Carlo.*