1. Suppose you have generated independent \( U[0, 1] \) random variables \( U_1, U_2, \ldots \). Give a formula or algorithm for generating a single random variable \( X \) where \( X \) has:

   a. the following probability density function:
      \[
      f(x) = \frac{1}{2x\sqrt{2\pi}} \exp\left(-\frac{(\log x - 2)^2}{8}\right).
      \]

   b. probability density function
      \[
      f(x) = \frac{1}{96}x^3 e^{-x/2}
      \]
      for \( 0 \leq x \), otherwise \( f(x) = 0 \).

   c. A discrete random variable with probability function
      \[
      P[X = x] = \frac{1}{kx} (0.9)^x, x = 1, 2, \ldots, \text{ where } k = 2.3025.
      \]
      How many uniform random numbers are required on average to generate a single random variable with this distribution?

   d. What is the probability density function of the random variables \( Z \) generated by the following Matlab code?
      \[
      U = \text{rand}(1,100000);
      V = \text{rand}(1,100000);
      X = U \cdot (V < 1-U);
      W = \text{rand}(1,\text{length}(X));
      Z = \text{min}(W.\text{power}(1/2), X);
      \]
2. Define the function whose graph appears in Figure 1;

\[ f(x) = \frac{x^2}{1 + e^{-5x}} \]

and suppose we wish to estimate the integral

\[ O = \int_0^1 f(x)2x \, dx \]

using Monte Carlo integration. Give estimators based on a sequence of independent uniform [0,1] random variables \( U_1, \ldots, U_n \) of this integral using the following methods. For each indicate how to assess their relative efficiency.

**a.** Importance sampling:

**b.** A control variate:

**c.** A stratified random sample with two strata, [0,0.75] and [0.75,1].

**d.** Guess the optimal sample sizes if we use the stratified sample with strata [0,0.75] and [0.75,1]. You may pretend that your function is equal to the one you used as a control variate in part (b).
3. A model for a financial time series \( S_t \) is written with stochastic differential equation in the form

\[
dS_t = rS_t dt + \sigma S_t^{0.5} dW_t
\]

for a Wiener (standard Brownian motion) process \( W_t \).

a. Give two methods for simulating \( S_2 \) starting with \( S_0 = 10, r = 0.05, \sigma = 0.2 \) and step size 1.

b. Suppose independent simulations are conducted at two points in order to estimate the rho of a derivative which has payoff at maturity \( T = 2 \) given by

\[
V(S_T) = (S_T - 10) \text{ if } 8 \leq S_T \leq 12
\]

otherwise \( V(S_T) = 0 \).

Is it possible to replicate this derivative using the stock, a risk-free account, ordinary European call options and digital options (having payoff equal to $1 if \( S_T \geq K \)) possibly with different strike prices?

c. We simulated the value of the discounted payoff \( e^{-2r} V(S_T) \) under two different circumstances:

**Method 1** using interest rate \( r = 0.045 \) and \( r = 0.055 \) and independent simulations, \( n = 100,000 \) at each of \( r = 0.045 \) and \( r = 0.055 \).

**Method 2** using interest rate \( r = 0.045 \) and \( r = 0.055 \) and common random numbers. The results of (ii) are as follows:

<table>
<thead>
<tr>
<th>Number of simulations</th>
<th>( r )</th>
<th>estimate of ( var(e^{-2r} V(S_T)) )</th>
<th>average( e^{-2r} V(S_T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>0.045</td>
<td>0.46</td>
<td>0.57</td>
</tr>
<tr>
<td>100,000</td>
<td>0.055</td>
<td>0.43</td>
<td>0.63</td>
</tr>
</tbody>
</table>

The correlation coefficient between the vector of estimators at the two different values of \( r \) using Method 2 was 0.8. Use this data to estimate rho and estimate the efficiency of the use of common random numbers relative to the use of independent simulations as in Method 1.