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Final Exam, Stat 901

December 14, 2001. 3 Hours

Instructor: D. L. McLeish

Do any SIX (6) of the questions below directly on the test paper.

1. Prove or disprove with a counter-example;

   (a) If a sequence of events A_k satisfy $A_k \subset A_{k+1}$..for all $k = 1, 2, ...$ then

   $$P(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n).$$

   (b) If X has a continuous c.d.f. $F(x)$ then the random variable $F(X)$ has a uniform distribution on the interval $[0, 1]$.

   (c) If X_i; $i = 1, 2, ...$ are random variables, then $Y_n = \sup\{X_m; m \geq n\}$ is a random variable for each n and converges to a random variable as $n \to \infty$.

2. Prove or disprove with a counter-example;

(a) If $|X|^p$ is integrable for $p \geq 1$, then for any constant $\epsilon > 0$,

$$P[|X| \geq \epsilon] \leq \frac{E|X|^p}{\epsilon^p}$$

(b) For every integrable random variable X and every $\epsilon > 0$,

$$P[|X| \geq \epsilon] < \frac{E|X|}{\epsilon}$$

(c) For any value of $t > 0$ and random variable X with moment generating function $m_X(t)$,

$$P[X > c] \leq e^{-tc} m_X(t)$$

3. Prove or disprove with a counter-example;

(a) If $X_1, X_2, ..., X_n$ are independent $N(0,1)$ random variables and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \to 1 \text{ almost surely as } n \to \infty.$$

(b) Define $M_n = median\ (X_1, X_2, \ldots, X_n)$. Then $M_n \to 0$ almost surely as $n \to \infty$.

4. Prove or disprove with a counter-example;

(a) If F_n, F, G_n, G are all cumulative distribution functions and $F_n \Rightarrow F$ and $G_n \Rightarrow G$, then $F_n * G_n \Rightarrow F * G$.

(b) If X_λ is a random variable with

$$P[X_\lambda = j] = \frac{\lambda^j e^{-\lambda}}{j!}, j = 0, 1, 2, ...$$

then the characteristic function of X_λ is

$$\varphi(t) = \exp(\lambda(e^{it} - 1))$$

(c) As $\lambda \to \infty$, the distribution of

$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}}$$

approaches the standard normal distribution.

5. Assume X is a random variable on the probability space $(\Omega, \mathcal{F}, P)$ with $E(X^2) < \infty$. If $\mathcal{H}$ is a sigma algebra with $\mathcal{H} \subset \mathcal{F}$

   (a) Define $E(X|\mathcal{H})$.

   (b) Prove for constants c, d that $E(cX + d|\mathcal{H}) = cE(X|\mathcal{H}) + d$.

   (c) Prove if $\mathcal{H} \subset \mathcal{G}$ are sigma-algebras, $E[E(X|\mathcal{G})|\mathcal{H}] = E(X|\mathcal{H})$. Does the same hold if $\mathcal{G} \subset \mathcal{H}$?

6. Prove or disprove with a counter-example;

(a) Assume X, Y, Z are random variables on the probability space $(\Omega, \mathcal{F}, P)$, X is integrable and Y is independent of (X, Z). Then.

$$E[X|Y, Z] = E[X|Z] \quad \text{a.s.}$$

(b) If τ is an optional stopping time taking values in the set $\{1, 2, ..., n\}$ and $\{(X_t, \mathcal{H}_t); t = 1, 2, ..., n\}$ is a martingale, then

$$E[(X_{j+1} - X_j)I(\tau > j)|\mathcal{H}_j] = 0 \text{ a.s. for all } j = 1, ..., n-1.$$

(c) $E(X_\tau) = E(X_1)$.

7. Prove or disprove with a counter-example ANY THREE of the following statements.

(a) Assume the $X_i, i = 1, 2, ...$ are independent and identically distributed random variables with mean μ and finite variance σ^2. Define $Z_{2n} = \sum_{i=1}^{n} X_i - \sum_{i=n+1}^{2n} X_i$. Then $n^{-1/2} Z_{2n}$ converges weakly to a normal distribution..

(b) The characteristic function $\varphi(t)$ of every probability distribution is continuous at $t = 0$.

(c) If X_n converges almost surely to a random variable X then X_n converges in probability to X.

(d) If X_n converges weakly (in distribution) to the constant c then it converges in probability to c.