Chapter 8

Martingales in Continuous Time

We denote the value of *(continuous time) stochastic process* X at time t denoted by $X(t)$ or by X_t as notational convenience requires. For each $t \in [0, \infty)$ let H_t be a sub sigma-field of H such that $H_s \subset H_t$ whenever $s \leq t$. We call such a sequence a *filtration.* A stochastic process X is said to be *adapted* to the filtration if $X(t)$ is measurable H_t for all $t \in [0, \infty)$.

We assume the existence of a filtration and that all stochastic processes under consideration are adapted to that filtration H_t. We also assume that the filtration H_t is *right continuous*, i.e. that

$$\bigcap_{\epsilon>0} H_{t+\epsilon} = H_t. \tag{8.1}$$

Without loss of generality, we can assume that a filtration is right continuous because if H_t is any filtration, then we can make it right continuous by replacing it with

$$H_{t+} = \bigcap_{\epsilon>0} H_{t+\epsilon}. \tag{8.2}$$

We use the fact that the intersection of sigma fields is a sigma field. Note that any process that was adapted to the original filtration is also adapted to the new filtration H_{t+}. We also typically assume, by analogy to the definition of the Lebesgue measureable sets, that if A is any set with $P(A) = 0$, then $A \in H_0$. These two conditions, that the filtration is right continuous and contains the $P-$null sets are referred to as the *standard conditions.*

If s and t are two time points in $[0\ ,\ \infty)$ then we shall let $s \wedge t$ be the minimum of s and t.

Definition 161 *Let $X(t)$ be a continuous time stochastic process adapted to a right continuous filtration H_t, where $0 \leq t < \infty$. We say that X is a* martingale *if* $E|X(t)| < \infty$ *for all* t *and*

$$E[X(t)|H_s] = X(s) \tag{8.3}$$

for all $s < t$. *The process* $X(t)$ *is said to be a* submartingale *(respectively a* supermartingale*) if the equality is replaced by* $\geq$ *(respectively* $\leq$ *).*

Definition 162 *A random variable* τ *taking values in* $[0, \infty]$ *is a* stopping time *for a martingale* (X_t, H_t) *if for each* $t \geq 0$, $[\tau \leq t] \in H_t$.

Definition 163 *A set of random variables* $\{X_t; t \geq 0\}$ *is uniformly integrable if, for all* $\epsilon > 0$, *there is a* $c < \infty$ *such that* $E(|X_t|I(|X_t| > c)) < \epsilon$ *for all* $t \geq 0$.

If a sequence of random variables converges in probability or almost surely and it is uniformly integrable, then the sequence also converges in L^1.

Lemma 164 *Suppose there exists a function* $\phi(x)$ *such that* $\lim_{x\to\infty} \phi(x)/x = \infty$ *and* $E\phi(|X_t|) \leq B < \infty$ *for all* $t \geq 0$. *Then the set of random variables* $\{X_t; t \geq 0\}$ *is uniformly integrable.*

Lemma 165 *If* X *is an integrable random variable, then there exists a convex function* $\phi(x)$ *such that* $\lim_{x\to\infty} \phi(x)/x = \infty$ *and* $E(\phi(|X|) < \infty$.

The combination of the two lemmas above contains the Lebesgue dominated convergence theorem.

Lemma 166 *Let* (M_t, H_t) *be a (right-)continuous martingale and assume that the filtration satisfies the standard conditions. Then for any fixed* $T < \infty$, *the set of random variables* $\{M_t, t \leq T\}$ *is uniformly integrable.*

Theorem 167 *Let* (M_t, H_t) *be a (right-)continuous martingale and assume that the filtration satisfies the standard conditions. If* τ *is a stopping time, then the process*

$$X_t = M_{t\wedge\tau}$$

is also a continuous martingale with respect to the same filtration.

Theorem 168 *(Doob's* L^p *Inequality) Suppose* (M_t, H_t) *is a (right-)continuous non-negative submartingale and put* $M_T^* = \sup_{0\leq t\leq T} M_t$. *Then for* $p \geq 1$, *and all* T

$$\lambda^p P[M_T^* > \lambda] \leq E[M_T^p] \text{ and}$$
$$||M_T^*||_p \leq \frac{p}{p-1}||M_T||_p, \quad \text{if } p > 1$$

Theorem 169 *(Martingale Convergence Theorem) Suppose a (right-)continuous martingale* M_t *satisfies* $\sup_t E(|M_t|^p) < \infty$ *for some* $p \geq 1$. *Then there exists a random variable* M_∞ *such that* $M_t \to M_\infty$ *a.s. If* $p > 1$, *then the convergence also holds in* L^p.

8.1 The Brownian Motion Process

The single most important continuous time process in the construction of financial models is the Brownian motion process. A Brownian motion is the oldest continuous time model used in finance and goes back to Bachelier around the turn of the last century. It is also the most common building block for more sophisticated continuous time models called diffusion processes.

The Brownian motion process is a random continuous time process $W(t)$ defined for $t \geq 0$ such that $W(0)$ takes some predetermined value, usually 0, and for each $0 \leq s < t$, $W(t) - W(s)$ has a normal distribution with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$. The parameters μ and σ are the drift and the diffusion parameters of the Brownian motion and the special case $\mu = 0, \sigma = 1$, $W(t)$ is often referred to as a standard Brownian motion or a Wiener process. Further properties of the Brownian motion process that are important are:

A Brownian motion process exists such that the sample paths are each continuous functions (with probability one)

The joint distribution of any finite number of increments $W(t_2)-W(t_1), W(t_4)-W(t_3), ...W(t_k) - W(t_{k-1})$ are independent normal random variables for $0 \leq t_1 \leq t_2 ... \leq t_k$.

Some further properties of the (standard) Brownian Motion Process.

1. $Cov(W(t), W(s)) = \min(s, t)$
2. If a Gaussian process has $E(X_t) = 0$ and $Cov(X(t), X(s)) = \min(s, t)$, then it has independent increments. If it has continuous sample paths and if $X_0 = 0$, then it is standard Brownian motion.

Define the triangular function

$$\Delta(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2(1-t) & \text{for } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and similar functions with base of length 2^{-j}

$$\Delta_{j,k}(t) = \Delta(2^j t - k) \quad \text{for } j = 1, 2, ...\text{and } k = 0, 1, ..., 2^j - 1.$$
$$\Delta_{0,0}(t) = 2t, \qquad 0 \leq t \leq 1$$

Theorem 170 *(Wavelet construction of Brownian motion) Suppose the random variables $Z_{j,k}$ are independent $N(0,1)$ random variables. Then series* *below converges uniformly (a.s.) to a Standard Brownian motion process $B(t)$ on the interval* $[0,1]$.

$$B(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} 2^{-j/2-1} Z_{j,k} \Delta_{j,k}(t)$$

The standard Brownian motion process can be extended to the whole interval $[0,\infty)$ by generating independent Brownian motion

processes $B^{(n)}$ on the interval $[0,1]$ and defining $W(t) = \sum_{j=1}^{n} B^{(j)}(1) + B^{(n+1)}(t-n)$ whenever $n \le t < n+1$.

Theorem 171 *If $W(t)$ is a standard Brownian motion process on $[0,\infty)$, then so are the processes $X_t = \frac{1}{\sqrt{a}} W(at)$ and $Y_t = tW(1/t)$ for any $a > 0$.*

Example 172 *(Examples of continuous martingales) Let W_t be a standard Brownian motion process. Then the processes*

1. W_t
2. $X_t = W_t^2 - t$
3. $\exp(\alpha W_t - \alpha^2 t/2)$, α any real number

are all continuous martingales

Theorem 173 *(Ruin probabilities for Brownian motion) If $W(t)$ is a standard Brownian motion and the stopping time τ is defined by*

$$\tau = \inf\{t; W(t) = -B \text{ or } A\}$$

where A and B are positive numbers, then $P(\tau < \infty) = 1$ and

$$P[W_\tau = A) = \frac{B}{A+B}$$

Theorem 174 *(Hitting times) If $W(t)$ is a standard Brownian motion and the stopping time τ is defined by*

$$\tau_a = \inf\{t; W(t) = a\}$$

where $a > 0$, then

1. $P(\tau_a < \infty) = 1$
2. *τ_a has a Laplace Transform given by*

$$E(e^{-\lambda \tau_a}) = e^{-\sqrt{2\lambda}|a|}.$$

3. *The probability density function of* τ_a *is*

$$f(t) = \frac{a}{t^{3/2}}\phi(a/\sqrt{t})$$

where ϕ *is the standard normal probability density function.*

4. *The cumulative distribution function is given by*

$$P[\tau_a \le t] = 2P[B(t) > a] = 2[1 - \Phi(\frac{a}{\sqrt{t}})].$$

5. $E(\tau_a) = \infty$

Corollary 175 *If* $B_t^* = \max\{B(s); 0 < s < t\}$ *then for* $a \ge 0$,

$$P[B_t^* > a] = P[\tau_a \le t] = 2P[B(t) > a]$$

Proposition 176 *(Reflection&Strong Markov Property) If* τ *is a stopping time with respect to the usual filtration of a standard Brownian motion* $B(t)$, *then the process*

$$\widetilde{B}(t) = \begin{cases} B(t) & t < \tau \\ 2B(\tau) - B(t) & t \ge \tau \end{cases}$$

is a standard Brownian motion.

Proposition 177 *(Last return to 0) Consider the random time* $L = \sup\{t \le 1; B(t) = 0\}$. *Then* L *has c.d.f.*

$$P[L \le s] = \frac{2}{\pi}\arcsin(\sqrt{s}), 0 < s < 1$$

and corresponding probability density function

$$\frac{d}{ds}\frac{2}{\pi}\arcsin(\sqrt{s}) = \frac{1}{\pi\sqrt{s(1-s)}}, 0 < s < 1$$

The Ito Integral

8.2 Introduction to Stochastic Integrals

The stochastic integral arose from attempts to use the techniques of Riemann-Stieltjes integration for stochastic processes. However, Riemann integration requires that the integrating function have *locally bounded variation* in order that the Riemann-Stieltjes sum converge. A function is said to have locally bounded variation if it can be written as the difference of two increasing processes. If the increasing processes are bounded then we say that their difference has finite variation. By constrast, many stochastic processes do not have paths of bounded variation. Consider, for example, a hypothetical integral of the form

$$\int_0^T f dW$$

where f is a nonrandom function of $t \in [0, T]$ and W is a standard Brownian motion. The Riemann-Stieljes sum for this integral would be

$$\sum_{i=1}^{n} f(s_i)[W(t_i) - W(t_{i-1})]$$

where $0 = t_0 < t_1 < t_2 < ... < t_n = T$ is a partition of $[0,\ T]$, and $t_{i-1} \le s_i \le t_i$. If we let the mesh of the partition go to zero then the Riemann-Stieljes sum will not converge because the Brownian motion paths are not of bounded variation. When f has bounded variation, we can circumvent this difficulty by formally defining the integral using integration by parts. Thus if we formally write

$$\int_0^T f dW = \left[fW - \int W df \right]_0^T$$

then the right hand side is well defined and can be used as the definition of the left hand side.

Integration by parts is too specialized for many applications. The integrand f is commonly replaced by some function of W or another stochastic process and is itself often not of bounded variation. Moreover, application of integration by parts can lead to difficulties. For example, integration by parts to evaluate the integral

$$\int_0^T W dW$$

leads to $\int_0^T W dW = W^2(T)/2$. Consider for a moment the possible range of limiting values of the Riemann Stieltjes sums

$$I_\alpha = \sum_{i=1}^{n} f(s_i)[W(t_i) - W(t_{i-1})].$$

where $s_i = t_{i-1} + \alpha(t_i - t_{i-1})$ for some $0 \le \alpha \le 1$. If the Riemann integral were well defined, then $I_1 - I_0 \to 0$ in probability. However when $f(s) = W(s)$, this difference

$$I_1 - I_0 = \sum_{i=1}^{n} [W(t_i) - W(t_{i-1})]^2 \to T$$
$$I_\alpha - I_0 \to \alpha T$$

The Ito stochastic integral corresponds to $\alpha = 0$ and approximates the integral with partial sums of the form

$$\sum_{i=1}^{n} W(t_{i-1})[W(t_i) - W(t_{i-1})]$$

the limit of which is, as the mesh size decreases, $\frac{1}{2}(W^2(T) - T)$ whereas evaluating the integrand at the right end point of the interval (i.e. taking $\alpha = 1$) results in $\frac{1}{2}(W^2(T) + T)$. Another natural choice is $\alpha = 1/2$ (called the Stratonovich integral) and note that this definition gives the answer $W^2(T)/2$ which is the same result obtained from the usual Riemann integration by parts. Which definition is "correct"? The Stratonovich integral has the advantage that it satisfies most of the traditional rules of deterministic calculus, for example **if the integral below is a Stratonovich integral,**

$$\int_0^T \exp(W_t) dW_t = \exp(W_T) - 1$$

While all definitions of a stochastic integral are useful, the main applications in finance are those in which the function $f(s)$ are the weights on various investments in a portfolio and the increment $[W(t_i) - W(t_{i-1})]$ represents the changes in price of the components of that portfolio over the next interval of time. Obviously one must commit to ones investments *before* observing the changes in the values of those investments. For this reason the Ito integral ($\alpha = 0$) seems the most natural in this context.

The Ito definition of a stochastic integral interprets the integral as a *linear isometry* from a Hilbert space of predictable processes into the Hilbert space of random variables. Notice that a stochastic integral

$$\int_0^T f(\omega, t) dW(t)$$

maps a function f on the product space $\Omega \times [0, T]$ into a space of random variables. Of course we need to apply some measurability conditions on the function f, and we will require two conditions below which permit a definition of the integral. This mapping $\int \ dW$ is said to be a *linear isometry* if it is a linear mapping (so $\int (f + g) dW = \int f dW + \int g dW$) and it preserves inner products. By this we mean that

$$E\{\int_0^T f(\omega, t) g(\omega, t) dt\} = E\{\int_0^T f(\omega, t) dW(t) \int_0^T g(\omega, t) dW(t)\}.$$

The inner product on the right hand side is the usual $L^2(P)$ inner product between random variables. That on the left hand side is defined as the integral of the product of the two functions over the product space $\Omega \times [0, T]$.

We now define the class of functions f to which this integral will apply. We assume that H_t is a standard Brownian filtration and that the interval $[0, T]$ is endowed with its Borel sigma field. Let $\mathcal{H}^2$ be the set of functions $f(\omega, t)$ on the product space $\Omega \times [0, T]$ such that

1. f is measurable with respect to the product sigma field on $\Omega \times [0, T]$.
2. For each $t \in [0, T]$, $f(., t)$ is measurable H_t. (in other words the stochastic process $f(., t)$ is adapted to H_t.
3. $E[\int_0^T f^2(\omega, t)dt] < \infty$.

The set of processes $\mathcal{H}^2$ is the natural domain of the Ito integral. However, before we define the stochastic integral on $\mathcal{H}^2$ we need to define it in the obvious way on the subclass of step functions in $\mathcal{H}^2$. Let $\mathcal{H}_0^2$ be the subset of $\mathcal{H}^2$ consisting of functions of the form

$$f(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega)\mathbf{1}(t_i < t \leq t_{i+1})$$

where the random variables a_i are measurable with respect to H_{t_i} and $0 = t_0 < t_1 < ... < t_n = T$. For such functions, the stochastic integral has only one natural definition:

$$\int_0^T f(\omega, t)dW(t) = \sum_{i=0}^{n-1} a_i(\omega)(\mathbf{W}(\mathbf{t}_{i+1)} - W(t_i))$$

and note that considered as a function of T, this forms a continuous time L^2 martingale.

Theorem 178 *For functions f and g in $\mathcal{H}_0^2$,*

$$E\{\int_0^T f(\omega, t)g(\omega, t)dt\} = E\{\int_0^T f(\omega, t)dW(t) \int_0^T g(\omega, t)dW(t)\}.$$

and

$$E\{\int_0^T f^2(\omega, t)dt\} = E(\int_0^T f(\omega, t)dW(t))^2 \tag{8.4}$$

These identities establish the isometry at least for functions in $\mathcal{H}_0^2$. The norm on stochastic integrals defined by

$$|| \int f dW||_{L(P)}^2 = E(\int_0^T f(\omega, t)dW(t))^2$$

agrees with the usual L^2 norm on the space of random functions

$$||f||^2 = E\{\int_0^T f^2(\omega, t)dt\}.$$

For this section, we will continue using the notation $||f||^2 = E\{\int_0^T f^2(\omega, t)dt\}$.

Lemma 179 *(Approximation lemma) For any $f \in \mathcal{H}^2$, there exists a sequence $f_n \in \mathcal{H}_0^2$ such that*

$$||f - f_n||^2 \to 0$$

The construction of a suitable approximating sequence f_n is easy. In fact we can construct a mesh $t_i = \frac{i}{2^n}T$ for $i = 0, 1, ...2^n - 1$ and define

$$f_n(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega)\mathbf{1}(t_i < t \le t_{i+1}) \tag{8.5}$$

with

$$a_i(\omega) = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(\omega, s)ds$$

the average of the function over the previous interval. This mapping from f to f_n we will denote later as $f_n = A_n(f)$ since it is linear and a contraction in the sense that $||A_n(f)|| \le ||f||$. Proving convergence of $A_n(f)$ to f is done by first proving that $||f - g_n|| \to 0$ where g_n is defined similarly but using the average of the function over the current interval. The proof follows from the next two lemmas.

Lemma 180 *Assume $f \in \mathcal{H}^2$ and f is bounded so that $|f(\omega, t)| < B < \infty$ for all $\omega \in \Omega$ and $0 \le t \le T$. Define $t_i = \frac{i}{2^n}T$ for $i = 0, 1, ...2^n - 1$ and*

$$g_n(\omega, t) = \sum_{i=0}^{n-1} b_i(\omega)\mathbf{1}(t_i < t \le t_{i+1}) \tag{8.6}$$

where

$$b_i(\omega) = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(\omega, s)ds.$$

Then $||f - g_n|| \to 0$ as $n \to \infty$.

Lemma 181 *Suppose g_m is of the form (8.6). Then*

$$||A_n(g_m) - g_m|| \to 0 \quad \text{as } n \to \infty$$

where A_n is the mapping described in (8.5).

The definition of a stochastic integral for any $f \in \mathcal{H}^2$ is now clear from the approximation lemma above. Choose a sequence $f_n \in \mathcal{H}_0^2$ such that $||f - f_n||^2 \to 0$. Since the sequence f_n is Cauchy, the isometry property (8.4) shows that the stochastic integrals $\int_0^T f_n dW$ also forms a Cauchy sequence in $L_2(P)$. Since this space is complete (in the sense that Cauchy sequences converge to a random variable in the space), we can define $\int f dW$ to be the limit of the sequence $\int_0^T f_n dW$ as $n \to \infty$.

Proposition 182 *The integral is well-defined. i.e. if* $f_n \in \mathcal{H}_0^2$ *and* $f_n' \in \mathcal{H}_0^2$ *such that* $||f - f_n||^2 \to 0$ *and* $||f - f_n'|| \to 0$, *then*

$$\lim_n \int f_n dW = \lim_n \int f_n' dW.$$

Proposition 183 *(Ito Isometry) For functions* f *and* g *in* $\mathcal{H}^2$,

$$E\{\int_0^T f(\omega,t)g(\omega,t)dt\} = E\{\int_0^T f(\omega,t)dW(t)\int_0^T g(\omega,t)dW(t)\}.$$

Theorem 184 *(Ito integral as a continuous martingale) For any* f *in* $\mathcal{H}^2$, *there exists a continuous martingale* X_t *adapted to the standard Brownian filtration* H_t *such that*

$$X_t = \int_0^T f(\omega,s)1(s \le t)dW(s) \quad \text{for all } t \le T.$$

This continuous martingale we will denote by $\int_0^t f dW$.

Example 185 *(*$\int W_t dW$*). Show that* $\int_0^T W_t dW = (W_T^2 - T)/2$ *(almost surely).*

8.3 Extending the Ito Integral to $\mathcal{L}^2_{LOC}$

Definition 186 *Let* $\mathcal{L}^2_{LOC}$ *be the set of functions* $f(\omega,t)$ *on the product space* $\Omega \times [0,T]$ *such that*

1. f is measurable with respect to the product sigma field on $\Omega \times [0,T]$.
2. For each $t \in [0,T]$, $f(.,t)$ is measurable H_t. (in other words the stochastic process $f(.,t)$ is adapted to H_t.
3. $P(\int_0^T f^2(\omega,s)ds < \infty) = 1\}$

Clearly this space includes $\mathcal{H}^2$ and arbitrary continuous functions of a Brownian motion.

Definition 187 *Let* ν_n *be an increasing sequence of stopping times such that*

1. $P[\nu_n = T$ for some $n] = 1$
2. The functions $f_n(\omega,t) = f(\omega,t)1(t \le \nu_n) \in \mathcal{H}^2$ for each n.

Then we call this sequence a **localizing sequence** for the function f.

Theorem 188 *For any function in* $\mathcal{L}^2_{LOC}$, *the sequence of stopping times*

$$\nu_n = \min(T, \inf\{s; \int_0^s f^2(\omega,t)dt \geq n\})$$

is a localizing sequence for f.

Definition 189 *For any function in* $\mathcal{L}^2_{LOC}$, *let* $f_n(\omega,t) = f(\omega,t)1(t \leq \nu_n)$ *and* $X_{n,t} = \int_0^t f_n(\omega, s\ )dW_s$ *where this is the version which is a continuous martingale. We define the Ito integral of* f

$$\int_0^t f(\omega,s)dW_s = \lim_{n\to\infty} X_{n,t}$$

Theorem 190 *The limit* $\lim_{n\to\infty} X_{n,t}$ *exists and is continuous.*

The proof requires several lemmas

Lemma 191 *Assume* $f \in \mathcal{H}^2$ *is bounded and for some stopping time* ν *we have* $f(\omega,t) = 0$ *almost surely on the set* $\{\omega; t \leq \nu(\omega)\}$. *Then*

$$\int_0^t f(\omega,s)dB_s = 0 \quad \textit{almost surely on } \{\omega; t \leq \nu(\omega)\}.$$

Proof. Note that there is a bounded sequence $f_n \in \mathcal{H}^2_0$ such that $f_n \to f$ (this means $||f_n - f|| \to 0$ where the norm is given by $||f||^2 = E\{\int_0^T f^2(\omega,t)dt\}$. It follows that $f_n 1(t \leq \nu) \to f1(t \leq \nu)$. Write

$$f_n = \sum a_i 1(t_i < t \leq t_{i+1})$$

and

$$\hat{f}_n = \sum a_i 1(t_i \leq \nu)1(t_i < t \leq t_{i+1}) \in \mathcal{H}^2_0$$

The proof follows the following steps: THIS LEMMA IS CURRENTLY UNDER REPAIR)

1. $||\hat{f}_n(t) - f_n(t)1(t \leq \nu)|| \to 0$
2. Since $f_n(t)1(t \leq \nu) \to f(t)1(t \leq \nu) = 0$ we have from 1 that $\hat{f}_n \to 0$.
3. Therefore by 1, $\int_0^T f_n(t)1(t \leq \nu)dW_t \to 0$

■

Lemma 192 *(persistence of identity) Assume* $f, g \in \mathcal{H}^2$ *and* ν *is a stopping time such that*

$$f(\omega,s) = g(\omega,s) \ \textit{almost surely on the set } \{\omega; t \leq \nu(\omega)\}.$$

Then $\int_0^t f(\omega,s)dB_s = \int_0^t g(\omega,s)dB_s$ *almost surely on the set* $\{\omega; t \leq \nu(\omega)\}$.

Lemma 193 *For $f \in \mathcal{H}^2$ and a localizing sequence ν_n, if we define $X_{n,t} = \int_0^t f(\omega,s)1(s \le \nu_n)dW_s$ to be the continuous martingale version of the integral, then for $m < n$, we have $X_{n,t} = X_{m,t}$ almost surely on the set $\{\omega; t \le \nu_m\}$.*

Lemma 194 *The definition of the integral does not depend on the localizing sequence.*

Lemma 195 *(persistence of identity in $\mathcal{L}^2_{LOC}$) Assume $f, g \in \mathcal{L}^2_{LOC}$ and ν is a stopping time such that*

$$f(\omega,s) = g(\omega,s) \text{ almost surely on the set } \{\omega; t \le \nu(\omega)\}.$$

Then $\int_0^t f(\omega,s)dB_s = \int_0^t g(\omega,s)dB_s$ almost surely on the set $\{\omega; t \le \nu(\omega)\}$.

Theorem 196 *Suppose f is a continuous non-random function and $t_i = iT/n, i = 0, 1, ..., n$. Then the Riemann sums*

$$\sum f(W_{t_i})(W_{t_{i+1}} - W_{t_i}) \to \int_0^T f(W_s)dW_s \text{ in probability.}$$

Theorem 197 *Suppose f is a continuous non-random function on $[0,T]$. Then the process*

$$X_t = \int_0^t f(s)dW_s$$

is a zero mean Gaussian process with $Cov(X_s, X_t) = \int_0^{\min(s,t)} f^2(u)du$. Moreover the Riemann sums

$$\sum f(s_i)(W_{t_{i+1}} - W_{t_i}) \to \int_0^T f(s)dW_s \text{ in probability}$$

for any $t_i \le s_i \le t_{t+1}$.

Theorem 198 *(time change to Brownian motion) Suppose $f(s)$ is a continuous non-random function on $[0,\infty)$ such that*

$$\int_0^\infty f^2(s)ds = \infty.$$

Define

$$\tau_t = \inf\{u; \int_0^u f^2(s)ds \ge t\}.$$

Then

$$Y_t = \int_0^{\tau_t} f(s)dW_s$$

is a standard Brownian motion.

Definition 199 *(local martingale) The process M_t is a local martingale with respect to the filtration H_t if there exists a non-decreasing sequence of stopping times $\tau_k \to \infty$ a.s. such that the processes*

$$M_t^{(k)} = M_{t\wedge\tau_k} - M_0$$

are martingales with respect to the same filtration.

Theorem 200 *If $f \in \mathcal{L}^2_{LOC}$, then there is a continuous local martingale X_t such that*

$$X_t = \int_0^t f(\omega, s)dW_s \quad \text{all } 0 \le t \le T \quad \text{almost surely.}$$

Theorem 201 *If M_t is a continuous local martingale with $M_0 = 0$, and if*

$$\tau = \inf\{t; M_t = A \quad or \ = -B\}$$

is finite with probability 1, then $E(M_\tau) = 0$ and

$$P[M_\tau = A] = \frac{B}{A+B}$$

Theorem 202 *If M_t is a local martingale and τ a stopping time, then $M_{t\wedge\tau}$ is a local martingale with respect to the same filtration.*

Theorem 203 *A bounded local martingale is a martingale.*

Theorem 204 *A non-negative local martingale X_t with $E(|X_0|) < \infty$ is a supermartingale. If $E(X_T) = E(X_0)$ it is a martingale.*

Definition 205 *For a stopping time τ define*

$$H_\tau = \{A \in \cup_T H_t; A \cap [\tau \le t] \in H_t, \quad for\ all\ t\}. \tag{8.7}$$

We have already proved that H_τ is a sigma-algebra.

Theorem 206 *If (X_t, H_t) is a bounded continuous martingale, τ is a stopping time, and if $A \in H_\tau$ then*

$$E(X_\tau 1_A 1_{\{\tau<s\}}) = E(X_s 1_A 1_{\{\tau<s\}})$$

Theorem 207 *If (X_t, H_t) is a bounded continuous martingale and $\nu \le \tau$ are stopping time, then*

$$E(X_\tau|H_\nu) = X_\nu \quad a.s.$$

Theorem 208 *If (X_t, H_t) is a bounded continuous martingale and τ a stopping times, then*

$$(X_{\tau\wedge t}, H_{\tau\wedge t}) \quad \text{is also a martingale.}$$

Theorem 209 *It τ_t is a non-decreasing family of stopping times, and X_t is a continuous local martingale, then the stopped sequence $X(\tau_t)$ is a local martingale.*

8.4 Ito's Formula

Introduce the differential notation

$$dX_t = g(t, W_t)dt + f(t, W_t)dW_t$$

to mean (this is its only possible meaning) the analogue of this equation written in integral form:

$$X_t = X_0 + \int_0^t g(s, W_s)ds + \int_0^t f(s, W_s)dW_s$$

where we assume that the functions g and f are such that these two integrals, one a regular Riemann integral and the other a stochastic integral, are well-defined.

Theorem 210 *(Ito's formula) For a function f with continuous second derivative,*

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

This result allows us to define the stochastic integral on the right side pathwise (i.e. for each ω). In other words

$$\int_0^t f'(W_s)dW_s = f(W_t) - f(0) - \frac{1}{2}\int_0^t f''(W_s)ds$$

Putting $f'(x) = x$ we obtain the previous result

$$\int_0^t W_s dW_s = \frac{W_t^2 - t}{2}.$$

Theorem 211 *(More general version:Ito's formula); Suppose $f(t, x)$ is once continously differentiable in t and twice in x. Denote its derivatives by*

$$f_1(t, x) = \frac{\partial f}{\partial t}, \qquad f_{22}(t, x) = \frac{\partial^2 f}{\partial x^2}, etc$$

Then

$$df(t, W_t) = \{f_1(t, W_t) + \frac{1}{2}f_{22}(t, W_t)\}dt + f_2(t, W_t)dW_t.$$

Corollary 212 *Suppose the function satisfies*

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}.$$

Then $f(t, X_t)$ is a local martingale and moreover if $E\int_0^T f_2^2(t, W_t)dt < \infty$, then it is a martingale.

The above condition $\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}$ is much like the heat equation usually written as

$$\frac{\partial f}{\partial t} = \lambda \frac{\partial^2 f}{\partial x^2} \text{ where } \dot{\lambda} > 0.$$

Indeed in the direction of reversed time this is a special case of the heat equation.

Example 213 *Consider $f(t,x) = tx - x^3/3$. Then $f(t, W_t)$ is a martingale and if we define a stopping time as*

$$\tau = \inf\{t; W_t = A \quad or \; W_t = -B\},$$

then $Cov(\tau, W_\tau) = \frac{1}{3}AB(A-B)$.

Example 214 *Consider*

$$f(t,x) = e^{\alpha x - \alpha^2 t/2}.$$

Then $M_t = f(t, W_t)$ is a martingale.

Theorem 215 *(Ruin probabilities: Brownian motion with drift)Let $X_0 = 0$ and*

$$dX_t = \mu dt + \sigma dW_t.$$

Define

$$\tau = \inf\{t; X_t = A \quad or \; X_t = -B\}.$$

Then

$$P(X_\tau = A) = \frac{\exp(-2\mu B/\sigma^2) - 1}{\exp(-2\mu(A+B)/\sigma^2) - 1}$$

This can be compared with the corresponding formula for the hitting probabilities for a biased random walk obtained earlier

$$p_A = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}, q = 1 - p$$

and these return exactly the same value if we use parameters satisfying $q/p = e^{-2\mu/\sigma^2}$ whereas if we choose a more natural choice of parameters for the Brownian motion approximating the random walk determined by the mean and variance per unit time $\mu = 2p - 1, \sigma^2 = 1 - (2p-1)^2$ then the hitting probabilities are extremely close provided that μ/σ^2 is small.

Theorem 216 *If $X_0 = 0$ and we define $M = \max_{0<t<\infty} X_t$ where*

$$dX_t = \mu dt + \sigma dW_t$$

then $M = \infty$ with probability 1 if $\mu \geq 0$ and otherwise M has an exponential distribution with mean $\sigma^2/(-2\mu)$.

Definition 217 *Suppose X_t is defined by X_0 and the stochastic differential equation*

$$dX_t = a(\omega, t)dt + b(\omega, t)dW_t.$$

By $\int_0^t f(\omega, s)dX_s$ we mean the integral

$$\int_0^t f(\omega, s)a(\omega, s)ds + \int_0^t f(\omega, s)b(\omega, s)dW_s$$

provided that these integrals are well defined.

Theorem 218 *(Ito's formula the third). Suppose X_t satisfies $dX_t = a(\omega, t)dt + b(\omega, t)dW_t$. Then for any function f such that f_1 and f_{22} are continuous,*

$$\begin{aligned} df(t, X_t) &= f_1(t, X_t)dt + f_2(t, X_t)dX_t + \frac{1}{2}f_{22}(t, X_t)dX_t \cdot dX_t \\ &= (a(\omega, t)f_2(t, X_t) + f_1(t, X_t) + \frac{1}{2}f_{22}(t, X_t)b^2(\omega, t))dt + f_2(t, X_t)b(\omega, t)dW_t \end{aligned}$$

Summary 219 *(Rules of box Algebra)*

$$\begin{aligned} dt \cdot dt &= 0 \\ dt \cdot dW_t &= 0 \\ dW_t \cdot dW_t &= dt \end{aligned}$$

Example 220 *(Geometric Brownian Motion) Suppose X_t satisfies*

$$dX_t = aX_t dt + \sigma X_t dW_t$$

Then Y_t=ln(X_t) is a Brownian motion with drift

$$dY_t = (a - \frac{\sigma^2}{2})dt + \sigma dW_t.$$

Example 221 *(Ito's formula for geometric Brownian motion). Suppose X_t is a geometric Brownian motion satisfying*

$$dX_t = aX_t dt + \sigma X_t dW_t.$$

Then for a function $f(t, x)$ with one continuous derivative with respect to t and two with respect to x, and $Y_t = f(t, X_t)$,

$$dY_t = \{f_1(t, X_t) + \frac{1}{2}f_{22}(t, X_t)\sigma^2 X_t^2\}dt + f_2(t, X_t)dX_t.$$

Definition 222 *A standard process X_t is one satisfying $X_0 = x_0$ and*

$$dX_t = a(\omega, t)dt + b(\omega, t)dW_t$$

where $\int_0^T \mid a(\omega, t)|dt < \infty$ and $\int_0^T \mid b(\omega, t)|^2 dt < \infty$ with probability one.

Theorem 223 *(Ito's formula: standard processes) If X_t is a standard process and $f(t,x)$ has one continuous derivative with respect to t and two with respect to x,*

$$\begin{aligned} df(t,X_t) &= f_1(t,X_t)dt + f_2(t,X_t)dX_t + \frac{1}{2}f_{22}(t,X_t)dX_t \cdot dX_t \\ &= (a(\omega,t)f_2(t,X_t) + f_1(t,X_t) + \frac{1}{2}f_{22}(t,X_t)b^2(\omega,t))dt + f_2(t,X_t)b(\omega,t)dW_t \end{aligned}$$

Theorem 224 *(Ito's formula for two processes) If*

$$\begin{aligned} dX_t &= a(\omega,t)dt + b(\omega,t)dW_t \\ dY_t &= \alpha(\omega,t)dt + \beta(\omega,t)dW_t \end{aligned}$$

then

$$\begin{aligned} df(X_t,Y_t) &= f_1(X_t,Y_t)dX_t + f_2(X_t,Y_t)dY_t + \\ &\frac{1}{2}f_{11}(X_t,Y_t)dX_t \cdot dX_t + \frac{1}{2}f_{22}(X_t,Y_t)dY_t \cdot dY_t \\ &+ f_{12}(X_t,Y_t)dX_t \cdot dY_t \\ &= f_1(X_t,Y_t)dX_t + f_2(X_t,Y_t)dY_t \\ &+ \frac{1}{2}f_{11}(X_t,Y_t)b^2dt + \frac{1}{2}f_{22}(X_t,Y_t)\beta^2dt \\ &+ f_{12}(X_t,Y_t)b\beta dt \end{aligned}$$

Theorem 225 *(the product rule) If*

$$\begin{aligned} dX_t &= a(\omega,t)dt + b(\omega,t)dW_t \\ dY_y &= \alpha(\omega,t)dt + \beta(\omega,t)dW_t \end{aligned}$$

then

$$d(X_tY_t) = Y_tdX_t + X_tdY_t + b\beta dt$$

This product rule reduces to the usual with either of β or b is identically 0.

Chapter 9

Review Problems.

1. Suppose that a measurable function $f(x)$ satisfies $\alpha \leq f(x) \leq \beta$ for constants α, β. Show that
$$\alpha\mu(A) \leq \int_A f(x)d\mu \leq \beta\mu(A)$$
for any measurable set A and measure μ.

2. Give an example of a sequence of simple functions $Y_n(\omega)$ which are increasing to the function $X(\omega) = \omega$ defined on $\Omega = [0,1]$ so that
$$\int_\Omega X(\omega)d\lambda = lim_{n\to\infty} \int_\Omega Y_n(\omega)d\lambda.$$

3. Suppose f is integrable with respect to Lebesgue measure on the real line. Show that
$$lim_{n\to\infty} \int_{[n,\infty)} f(x)d\lambda = 0$$

4. Consider a sequence of events A_n and define the random variable $Y_n = I_{A_n}$. When is $pY_n = 1$? When is it 0? Apply Fatou's lemma and the monotone convergence theorem to the sequence of random variables Y_n and determine what they imply for the sequence of events A_n.

5. Define a random variable as follows; Suppose Z has a standard normal distribution and X has a discrete uniform distribution on the set of points $\{0,1,2,3\}$. What is the cumulative distribution function of the random variable $min(Z,X)$? Is this random variable discrete, continuous, or of some other type?

6. Prove that for any non-negative random variable X with cumulative distribution function $F(x)$, if X has finite variance, then
$$lim\ \ sup_{x\ \to\infty} x^2(1-F(x)) < \infty.$$

(*Hint: Compare* $x^2 P[X > x]$ *with* $E(X^2)$.)

7. Suppose $X \sim bin(n, p)$. Find

$$\sum_{j=1}^{n} P[X \geq j].$$

8. If $F(x)$ is the standard normal cumulative distribution function, show that $x^k(1 - F(x)) \to 0$ as $x \to \infty$ for any $k < \infty$.

9. If X and Y are independent random variables, show that the characteristic function of $X + Y$ equals the product of the two characteristic functions.

10. If X and Y are dependent random variables with joint cumulative distribution function $F(x, y)$, show that for any $a < b$,

$$F(b, b) - F(a, b) - F(b, a) + F(a, a) \geq 0.$$

11. Suppose X has a normal distribution with expected value 0 and variance 1. Then show, using a theorem allowing interchange of limits and expectation, that

$$lim_{N\to\infty} \sum_{j=0}^{N} \frac{E(X^j)t^j}{j!} = m_X(t)$$

12. Show that if $|X|^p$, $p \geq 1$ is integrable, then

$$|E(X)|^p \leq E[|X|^p]$$

(a) Prove the weak law of large numbers in the following form:

If X_n are independent random variables with common mean $E(X_n) = \mu$, and common variance

$$var(X_n) = \sigma^2 < \infty,$$

prove $\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$ in probability.

(b) Define $Z_{2n} = \sum_{i=1}^{n} X_i - \sum_{i=n+1}^{2n} X_i$. Assume the X_i are identically distributed, show that Z_{2n} is asymptotically normally distributed.

(a) Prove that any characteristic function is non-negative definite:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \varphi(t_i - t_j) z_i \bar{z}_j \geq 0$$

for all real $t_1, ...t_n$ and complex $z_1, ..., z_n$.

(b) Prove that the characteristic function of any probability distribution is continuous at $t = 0$.

(a) Prove that if X_n converges with probability 1 to a random variable X then X_n converges weakly to X .

(b) Prove that if X_n converges weakly to the constant c then it converges in probability.

13. An urn has r_0 red balls and b_0 black ones. On the n'th draw, a ball is drawn at random and then replaced with 2 balls of the same colour. Let r_n and b_n be the number of red and black balls after the n'th draw.

    (a) Prove that $\frac{b_n}{r_n+b_n}$ converges almost surely as $n \to \infty$.

    (b) Let the limiting random variable be Y . What is $E(Y)$?

14. Prove or provide counterexamples:

    (a) For independent random variables: $\sigma(X, Y) = \sigma(X) \cup \sigma(Y)$.

    (b) For any random variables X, Y, Z , $E\{E(X|Y)|Z\} = E\{E(X|Z)|Y\}$ almost surely.

    (c) For X, Y independent, $E(X|Y) = E(X)$ almost surely.

    (d) Whenever a sequence of cumulative distribution functions F_n satisfy $F_n(x) \to F(x)$ for all x , F is a cumulative distribution function of a random variable.

15. Prove the following assertions concerning conditional expectation for square integrable random variables:

    (a) If X is $\mathcal{G}$-measurable, $E(X|\mathcal{G}) = X$.

    (b) If X independent of $\mathcal{G}$, $E(X|\mathcal{G}) = E(X)$.

    (c) For any square integrable $\mathcal{G}$-measurable Z, $E(ZX) = E[ZE(X|\mathcal{G})]$.

16. Prove that a characteristic function has the following four properties:

    (a) φ exists for any X .

    (b) $\varphi(0) = 1$.

    (c) $|\varphi(t)| \leq 1$ for all t .

    (d) φ is uniformly continuous.

17. Prove the following assertions:

    (a) convergence almost sure implies convergence in probability

    (b) convergence in probability implies weak convergence.

    (c) Weak convergence to a constant r.v. implies convergence in probability