Chapter 2

Measure Spaces

2.1 Families of Sets

Definition 7 *($\pi-$ systems) A family of subsets $\mathcal{F}$ of Ω is a $\pi-$system if, $A_k \in \mathcal{F}$ for $k = 1, 2$ implies $A_1 \cap A_2 \in \mathcal{F}$.*

A $\pi-$system is closed under finitely many intersections but not necessarily under unions. The simplest example of a $\pi-$system is the family of rectangles in Euclidean space. Clearly a Boolean algebra is a π-system but there are $\pi-$systems that are not Boolean algebras (see the problems).

Definition 8 *(Sigma-Algebra) $\mathcal{F}$ is sigma algebra if,*

(i) $A_k \in \mathcal{F}$ *for all* k *implies* $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$

(ii) $A \in \mathcal{F}$ *implies* $A^c \in \mathcal{F}$.

(iii) $\phi \in \mathcal{F}$.

Note that only the first property of a Boolean algebra has been changed-it is slightly strengthened. Any sigma algebra is automatically a Boolean algebra.

Theorem 9 *(Properties of a Sigma-Algebra) If $\mathcal{F}$ is a sigma algebra, then*

(iv) $\Omega \in \mathcal{F}$.

(v) $A_k \in \mathcal{F}$ *for all* k *implies* $\cap_{k=1}^{\infty} A_k \in \mathcal{F}$

Proof. Note that $\Omega = \varphi^c \in \mathcal{F}$ by properties (ii) and (iii). This verifies (iv). Also $\cap_{k=1}^{\infty} A_k = (\cup_{k=1}^{\infty} A_k^c)^c \in \mathcal{F}$ by properties (i) and (ii). ∎

Theorem 10 *(Intersection of sigma algebras) Let $\mathcal{F}_\lambda$ be sigma algebras for each $\lambda \in \Lambda$. The index set Λ may be finite or infinite, countable or uncountable. Then $\cap_\lambda \mathcal{F}_\lambda$ is a sigma-algebra.*

Proof. Clearly if $\mathcal{F} = \cap_\lambda \mathcal{F}_\lambda$ then $\varphi \in \mathcal{F}$ since $\varphi \in \mathcal{F}_\lambda$ for every λ. Similarly if $A \in \mathcal{F}$ then $A \in \mathcal{F}_\lambda$ for every λ and so is A^c. Consequently $A^c \in \mathcal{F}$. Finally if $A_n \in \mathcal{F}$ for all $n = 1, 2, ...$then $A_n \in \mathcal{F}_\lambda$ for every n, λ and $\cup_{n=1}^\infty A_n \in \mathcal{F}_\lambda$ for every λ. This implies $\cup_{n=1}^\infty A_n \in \mathcal{F}$. ■

Definition 11 *(sigma algebra generated by family of sets) If $\mathcal{C}$ is a family of sets, then the* sigma algebra generated by $\mathcal{C}$, *denoted $\sigma(\mathcal{C})$, is the intersection of all sigma-algebras containing $\mathcal{C}$. It is the smallest sigma algebra which contains all of the sets in $\mathcal{C}$.*

Example 12 *Consider $\Omega = [0, 1]$ and $\mathcal{C} = \{[0, .3], [.5, 1]\} = \{A_1, A_2\}$, say. Then $\sigma(\mathcal{C}) = \{\varphi, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3, \Omega\}$ where we define $A_3 = (.3, .5)$. (There are 8 sets in $\sigma(\mathcal{C})$).*

Example 13 *Define Ω to be the interval (0,1] and $\mathcal{F}_\prime$ to be the class of all sets of the form $(a_0, a_1] \cup (a_2, a_3] \cup ... \cup (a_{n-1}, a_n]$ where $0 \leq a_0 \leq ... \leq a_n \leq 1$. Then $\mathcal{F}_\prime$ is a Boolean algebra but not a sigma algebra.*

Example 14 *(all subsets) Define $\mathcal{F}_\prime$ to be the class of* all *subsets of any given set Ω. Is this a Boolean algebra? Sigma Algebra? How many distinct sets are there in $\mathcal{F}_\prime$ if Ω has a finite number, N points?*

Example 15 *A and B play a game until one wins once (and is declared winner of the match). The probability that A wins each game is* 0.3, *the probability that B wins each game is* 0.2 *and the probability of a draw on each game is* 0.5. *What is a suitable probability space, sigma algebra and the probability that A wins the match?*

Example 16 *(Borel Sigma Algebra) The Borel Sigma Algebra is defined on a topological space $(\Omega, \mathcal{O})$ and is $\mathcal{B} = \sigma(\mathcal{O})$.*

Theorem 17 *The Borel sigma algebra on $\mathcal{R}$ is $\sigma(\mathcal{C})$, the sigma algebra generated by each of the classes of sets $\mathcal{C}$ described below;*

1. $\mathcal{C}_1 = \{(a, b);\, a \leq b\}$
2. $\mathcal{C}_2 = \{(a, b];\, a \leq b\}$
3. $\mathcal{C}_3 = \{[a, b);\, a \leq b\}$
4. $\mathcal{C}_4 = \{[a, b];\, a \leq b\}$
5. $\mathcal{C}_5$ =the set of all open subsets of $\mathcal{R}$
6. $\mathcal{C}_6$ =the set of all closed subsets of $\mathcal{R}$

To prove the equivalence of 1 and 5 above, we need the following theorem which indicates that any open set can be constructed from a countable number of open intervals.

Theorem 18 *Any open subset of $\mathcal{R}$ is a countable union of open intervals of the form (a,b).*

Proof. Let O be the open set and $x \in O$. Consider the interval $I_x = \cup\{(a,b); a < x < b, (a,b) \subset O\}$. This is the largest open interval around x that is entirely contained in O. Note that if $x \neq y$, then $I_x = I_y$ or $I_x \cap I_y = \varphi$. This is clear because if there is some point $z \in I_x \cap I_y$, then $I_x \cup I_y$ is an open interval containing both x and y and so since they are, by definition, the largest such open interval, $I_x \cup I_y = I_x = I_y$. Then we can clearly write

$$\begin{aligned} O &= \cup\{I_x; x \in O\} \\ &= \cup\{I_x; x \in O, x \text{ is rational}\} \end{aligned}$$

since every interval I_x contains at least one rational number. ■

Definition 19 (Lim Sup, Lim Inf) *For an arbitrary sequence of events A_k*

$$\lim_{n\to\infty} \sup \mathrm{A}_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = [A_n \ i.o.]$$

$$\lim_{n\to\infty} \inf A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = [A_n \ a.b.f.o.]$$

The notation A_n i.o. refers to A_n infinitely often and A_n $a.b.f.o.$ refers to A_n "all but finitely often".

A given point ω is in $\lim_{n\to\infty} \sup \mathrm{A}_n$ if and only if it lies in infinitely many of the individual sets A_n. The point is in $\lim_{n\to\infty} \inf A_n$ if and only if it is in *all but a finite number of the sets.* Which of these two sets is bigger? Compare them with $\cup_{k=n}^{\infty} A_k$ and $\cap_{k=n}^{\infty} A_k$ for any fixed n. Can you think of any circumstances under which $\limsup A_n = \liminf A_n$? You should be able to prove that

$$[\limsup A_n]^c = \liminf A_n^c.$$

Theorem 20 *Assume $\mathcal{F}$ is a sigma-algebra. If each of $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$, then both $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$ and $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ are in $\mathcal{F}$.*

Definition 21 *(measurable space) A pair $(\Omega, \mathcal{F})$ where the former is a set and the latter a sigma algebra of subsets of Ω is called a* measurable space.

Definition 22 *(additive set function) Consider a space Ω and a family of subsets $\mathcal{F}_0$ of Ω such that $\phi \in \mathcal{F}_0$. Suppose μ_0 is a non-negative set function; i.e. has the properties that*

- $\mu_0 : \mathcal{F}_0 \to [0, \infty]$
- When F, G and $F \cup G \in \mathcal{F}_0$ and $F \cap G = \phi$, then $\mu_0(F) + \mu_0(G) = \mu_0(F \cup G)$.

Then we call μ_0 an *additive set function* on $(\Omega, \mathcal{F}_0)$.

Note that it follows that $\mu_0(\phi) = 0$ (except in the trivial case that $\mu_0(A) = \infty$ for every subset including the empty set. We rule this out in our definition of a measure.)

Definition 23 *We call* μ_0 *a* countably additive set function *on* $(\Omega, \mathcal{F}_0)$ *if, whenever all* A_n, $n = 1, 2, \ldots$ *are members of* $\mathcal{F}_0$ *and* $\cup_{n=1}^{\infty} A_n \in \mathcal{F}_0$ *, and the sets are disjoint (* $A_i \cap A_j = \phi$, $i \neq j$*) then it follows that*

$$\mu_0(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$$

We saw at the beginning of this chapter that the concept of a $\pi-$system provides one basic property of a Boolean algebra, but does not provide for unions. In order to insure that such a family is a $\sigma-$algebra we need the additional conditions provided by a $\lambda-$system (below).

Definition 24 *A family of events* $\mathcal{F}$ *is called a* λ*-system if the following conditions hold:*

1. $\Omega \in \mathcal{F}$
2. $A, B \in \mathcal{F}$ and $B \subset A$ implies $A \backslash B \in \mathcal{F}$
3. If $A_n \in \mathcal{F}$ for all $n = 1, 2, \ldots$ and $A_n \subset A_{n+1}$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$

A $\lambda-$system is closed under set differences if one set is included in the other and monotonically increasing countable unions. It turns out this this provides the axioms that are missing in the definition of a π-system to guarantee the conditions of a sigma-field are satisfied.

Proposition 25 *If* $\mathcal{F}$ *is both a* π*-system and a* λ*-system then it is a sigma-algebra.*

Proof. By the properties of a $\lambda-system$, we have that $\Omega \in \mathcal{F}$ and if $A \in \mathcal{F}$ then $A^c = \Omega \setminus A \in \mathcal{F}$. So we need only show that $\mathcal{F}$ is closed under countable unions. Note that since $\mathcal{F}$ is a $\pi-$system it is closed under finite intersections. Therefore if $A_n \in \mathcal{F}$ for each $n = 1, 2, \ldots$ then $B_n = \cup_{i=1}^{n} A_i = (\cap_{i=1}^{n} A_i^c)^c \in \mathcal{F}$ for each n and since $B_n \subset B_{n+1}$, $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n \in \mathcal{F}$ by the third property of a $\lambda - system$. ■

Theorem 26 *(The* $\pi - \lambda$ *Theorem) Suppose a family of sets* $\mathcal{F}$ *is a* $\pi-system$ *and* $\mathcal{F} \subset \mathcal{G}$ *where* $\mathcal{G}$ *is a* λ*-system. Then* $\sigma(\mathcal{F}) \subset \mathcal{G}$ *.*

This theorem is due to Dynkin and is proved by showing that the smallest $\lambda-$system containing $\mathcal{F}$ is a π-system and is therefore, by the theorem above, a sigma-algebra.

2.2 Measures

Definition 27 *(*measure*)* μ *is a (non-negative)* measure *on the measurable space* $(\Omega, \mathcal{F})$ *where* $\mathcal{F}$ *is a sigma-algebra of subsets of* Ω *if it is a countably additive (non-negative) set function* $\mu(); \mathcal{F} \to [0, \infty]$.

A measure μ satisfies the following conditions

(i) $\mu(A) \geq 0$ for all A.

(ii) If A_k disjoint, $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$

(iii) $\mu(\phi) = 0$

(iv) (monotone) $A \subset B$ implies $\mu(A) \leq \mu(B)$.

(v) (subadditive) $\mu(\cup_k A_k) \leq \sum_k \mu(A_k)$

(vi) (inclusion-exclusion). For finitely many sets,

$$\mu(\cup_{k=1}^{n} A_k) = \sum_k \mu(A_k) - \sum_{i<j} \mu(A_i \cap A_j) + ...$$

(vii) If A_k converges (i.e. is nested increasing or decreasing)

$$\mu(lim_n A_n) = lim_n \mu(A_n)$$
$$\text{where } \lim_n A_n = \begin{cases} \cup_n A_n \text{ if } A_n \text{ increasing} \\ \cap_n A_n \text{ if } A_n \text{ decreasing} \end{cases}$$

Definition 28 *(Measure space)The triple* $(\Omega, \mathcal{F}, \mu)$ *is called a* measure space.

Measures do exist which may take negative values as well but we leave discussion of these for later. Such measures we will call *signed measures*. For the present, however, we assume that every measure takes non-negative values only.

Definition 29 *(Probability measure) A* Probability measure *is a measure* P *satisfying* $P(\Omega) = 1$.

(Additional property) A probability measure also satisfies

(viii) $P(A^c) = 1 - P(A)$

Definition 30 *(Probability space) When the measure* P *is a probability measure, the triple* $(\Omega, \mathcal{F}, \mathcal{P})$ *is called a* probability space.

Theorem 31 *(Conditional Probability) For* $B \in \mathcal{F}$ *with* $P(B) > 0$, $Q(A) = P(A|B) = P(A \cap B)/P(B)$ *is a probability measure on the same space* $(\Omega, \mathcal{F})$.

2.3 Extending a measure from an algebra

Although measures generally need to be supported by sigma-algebras of sets, two probability measures are identical if they are identical on an algebra. The following Theorem is fundamental to this argument, and to the construction of Lebesgue measure on the real line.

Theorem 32 *(Caratheodory Extension) Suppose $\mathcal{F}_0$ is a (Boolean) algebra and μ_0 a countably additive set function from $\mathcal{F}_0$ into $[0,\infty]$. Then there is an extension of μ_0 to a measure μ defined on all of $\sigma(\mathcal{F}_0)$. Furthermore, if the total measure $\mu_0(\Omega)<\infty$ then the extension is unique.*

Proof. We do not provide a complete proof-details can be found in any measure theory text (e.g. Rosenthal, p.10-14.) Rather we give a short sketch of the proof. We begin by defining the outer measure of any set $E \subset \Omega$ (note it does not have to be in the algebra or sigma-algebra) by the smallest sum of the measures of sets in the algebra which cover the set E, i.e.

$$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \mu_0(A_n); E \subset \cup_{n=1}^{\infty} A_n, A_n \in \mathcal{F}_0\}.$$

Notice that the outer measure of a set in the algebra is the measure itself $\mu^*(E) = \mu_0(E)$ if $E \in \mathcal{F}_0$. Therefore, this outer measure is countably additive when restricted to the algebra $\mathcal{F}_0$. Generally, however, this outer measure is only subadditive; the measure of a countable union of disjoint events is less than or equal to the sum of the measures of the events. **If it were additive**, then it would satisfy the property;

$$\mu^*(E) = \mu^*(EQ) + \mu^*(EQ^c). \tag{2.1}$$

However, let us consider the class $\mathcal{F}$ of all sets Q for which the above equation (2.1) does hold. The rest of the work in the proof consists of showing that the class of sets $\mathcal{F}$ forms a sigma algebra and when restricted to this sigma algebra, the outer measure μ^* is countably additive, so is a measure. ■

The last condition in the extension theorem can be replaced by a weaker condition, that the measure is *sigma-finite.* In other words it suffices that we can write the whole space as a countable union of subsets A_i (i.e. $\Omega = \cup_{i=1}^{\infty} A_i$) each of which has finite measure $\mu_0(A_i) < \infty$. Lebesgue measure on the real line is sigma-finite but not finite.

Example 33 *Lebesgue measure Define $\mathcal{F}_0$ to be the set of all finite unions of intervals (open, closed or half and half) such as*

$$A = (a_0, a_1] \cup (a_2, a_3] \cup ... \cup (a_{n-1}, a_n]$$

*where $-\infty \le a_0 \le ... \le a_n \le \infty$. For A of the above form, define $\mu(A) = \sum_i (a_{2i+1} - a_{2i})$. Check that this is well-defined. Then there is a unique extension of this measure to all $\mathcal{B}$, the Borel subsets of $\mathcal{R}$. This is called the **Lebesgue measure.***

It should be noted that in the proof of Theorem 11, the sigma algebra $\mathcal{F}$ may in fact be a larger sigma algebra than $\sigma(\mathcal{F}_0)$ generated by the algebra. For example in the case of measures on the real line, we may take $\mathcal{F}_0$ to be all finite union of intervals. In this case $\sigma(\mathcal{F}_0)$ is the class of all Borel subsets of the real line but it is easy to check that $\mathcal{F}$ is a larger sigma algebra having the property of completeness, i.e. for any $A \in \mathcal{F}$ such that $\mu(A) = 0$, all subsets of A are also in $\mathcal{F}$ (and of course also have measure 0).

Example 34 *(the Cantor set) This example is useful for dispelling the notions that closed sets must either countain intervals or consist of a countable selection of points. Let $\Omega = [0,1]$ with P Lebesgue measure. Define $A_1 = \Omega \backslash \{(\frac{1}{3}, \frac{2}{3})\}$ and $A_2 = A\backslash\{(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})\}$ etc. In each case, $A_n$is obtained from A_{n-1} by deleting the open interval in the middle third of each interval in A_{n-1}. Define $A = \cap_{n=1}^{\infty} A_n$. Then A is a closed, uncountable set such that $P(A) = 0$ and A contains no nondegenerate intervals.*

2.4 Independence

Definition 35 *(Independent Events) A family of events $\mathcal{C}$ is (mutually) independent if*

$$P(A_{\lambda_1} \cap A_{\lambda_2} .... A_{\lambda_n}) \ = \ P(A_{\lambda_1}) P(A_{\lambda_2}) .... P(A_{\lambda_n}) \qquad (*)$$

for all n, $A_{\lambda_i} \in \mathcal{C}$ and for distinct λ_i.

Properties: Independent Events

1. A, B independent implies A, B^c independent.
2. Any A_λ can be replaced by A_λ^c in equation (*).

Definition 36 *Families of sigma-algebras $\{\mathcal{F}_\lambda;\ \lambda \in \Lambda\}$ are independent if for any $A_\lambda \in \mathcal{F}_\lambda$, the family of events$\{A_\lambda; \lambda \in \Lambda\}$ are mutually independent.*

Example 37 *(Pairwise independence does not imply independence) Two fair coins are tossed. Let A = first coin is heads, B = second coin is heads, C = we obtain exactly one heads. Then A is independent of B and A is independent of C but A, B, C are* ***not mutually independent****.*

2.4.1 The Borel Cantelli Lemmas

Clearly if events are individually too small, then there little or no probability that their lim sup will occur, i.e. that they will occur infinitely often.

Lemma 38 *For an arbitrary sequence of events A_n , $\sum_n P(A_n) < \infty$ implies $P[A_n \ i.o.] = 0$.*

Proof. Notice that

$$P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m) \leq P(\cup_{m=n}^{\infty} A_m) \leq \sum_{m=n}^{\infty} P(A_m) \text{ for each } n = 1, 2, ...$$

For any $\epsilon > 0$, since the series $\sum_{m=1}^{\infty} P(A_m)$ converges we can find a value of n sufficiently large that $\sum_{m=n}^{\infty} P(A_m) < \epsilon$. Therefore for every positive ϵ, $P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m) \leq \epsilon$ and so it must equal 0. ∎

The converse of this theorem is false without some additional conditions. For example suppose that Ω is the unit interval and the measure is Lebesgue. Define $A_n = [0, \frac{1}{n}], n = 1, 2, ...$ Now although $\sum P(A_n) = \infty$, it is still true that $P(A_n \text{ i.o.}) = 0$. However if we add the condition that the events are independent, we do have a converse as in the following.

Lemma 39 *For a sequence of* independent events A_n, $\sum_n P(A_n) = \infty$ *implies* $P[A_n \; i.o] = 1$.

Proof. We need to show that $P(A_n^c \quad \text{a.b.f.o.}) = 0$. This is

$$\begin{aligned} P(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m^c) &\leq \sum_{n=1}^{\infty} P(\cap_{m=n}^{\infty} A_m^c) \\ &\leq \sum_{n=1}^{\infty} \prod_{m=n}^{N_n} (1 - P(A_m)) \text{ for any sequence } N_n \\ &\leq \sum_{n=1}^{\infty} \exp\{-\sum_{m=n}^{N_n} P(A_m)\} \end{aligned}$$

where we have used the inequality $(1 - P(A_m)) \leq \exp(-P(A_m))$. Now if the series $\sum_{m=1}^{\infty} P(A_m)$ diverges to ∞ then we can choose the sequence N_n so that $\sum_{m=n}^{N_n} P(A_m) > nln2 - ln\epsilon$ in which case the right hand side above is less than or equal to ϵ. Since this holds for arbitrary $\epsilon > 0$, this verifies that $P(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m^c) = 0$. ∎

Definition 40 *(Almost surely)A statement* S *about the points in* Ω *holds almost surely(a.s.) or with probability one if the set of* ω *such that the statement holds has probability one. Thus Lemma 13 above states that* A_n *occurs infinitely often almost surely (a.s.) and Lemma 12 that* A_n^c *occurs all but finitely often (a.s.).*

2.4.2 Kolmogorov's Zero-one law

For independent events A_n, put

$$\mathcal{F} = \cap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, ...)$$

(call this the *tail sigma-algebra*). Events that are determined by the sequence $\{A_1, A_2, ...\}$ but not by a finite number such as $\{A_1, ...A_N\}$ are in the tail sigma-algebra. This includes events such as $[\limsup A_n], [\liminf A_n]$, $[\limsup A_{2^n}]$, etc.

Theorem 41 ***(zero-one law)*** *Any event in the tail sigma-algebra* $\mathcal{F}$ *has probability either 0 or 1.*

Proof. Define $\mathcal{F}_n = \sigma(A_1, A_2, ..., A_n)$ and suppose $B \in \mathcal{F}_n$ for fixed n. Then B is independent of $\mathcal{F}$ because it is independent of all sets in the larger sigma algebra $\sigma(A_{n+1}, A_{n+2}, ...)$. This means that every set $A \in \mathcal{F}$ is independent of every set in each $\mathcal{F}_n$ and therefore A is independent of each member of the Boolean Algebra of sets $\cup_{n=1}^{\infty}\mathcal{F}_n$. Therefore A is independent of $\sigma(\cup_{n=1}^{\infty}\mathcal{F}_n)$. But since

$$\cap_{n=1}^{\infty}\sigma(A_n, X_{n+1}, ...) \subset \sigma(\cup_{n=1}^{\infty}\mathcal{F}_n)$$

A is independent of itself, implying it has probability either 0 or 1 (see problem 18). ■

2.5 Problems.

1. Give an example of a family of subsets of the set $\{1, 2, 3, 4\}$ that is a $\pi-$system but NOT a Boolean algebra of sets.

2. Consider the space $\Re^2$and define the family of all rectangles with sides parallel to the axes. Show that this family is a π-system.

3. Let Ω be the real line and let $\mathcal{F}_n$ be the sigma-algebra generated by the subsets
$$[0, 1), [1, 2), ..., [n-1, n)$$
Show that the sigma-algebras are *nested* in the sense that $\mathcal{F}_1 \subset \mathcal{F}_2$. How do you know if a given set is in $\mathcal{F}_n$? Show that $\cup_{n=1}^{100}\mathcal{F}_n$ is a sigma-algebra.

4. As above, let Ω be the real line and let $\mathcal{F}_n$ be the sigma-algebra generated by the subsets
$$[0, 1), [1, 2), ..., [n-1, n)$$
Show that $\cup_{n=1}^{\infty} \mathcal{F}_n$ is not a sigma-algebra.

5. How do we characterise the open subsets of the real line $\Re$? Show that the Borel sigma algebra is also generated by all sets of the form $(\infty, x], x \in \Re$.

6. For an arbitrary sequence of events A_k , give a formula for the event $B_k =$ [the first of the A_j's to occur is A_k].

7. Write in set-theoretic terms the event that exactly two of the events A_1, A_2, A_3, A_4, A_5 occur.

8. Prove that if A_k is a nested sequence of sets (increasing or decreasing), then $\limsup A_n = \liminf A_n$ and both have probability equal to $lim_n P(A_n)$.

9. Prove Bayes Rule:

   If $P(\cup_n B_n) = 1$ for a disjoint finite or countable sequence of events B_n all with positive probability, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_n P(A|B_n)P(B_n)}$$

10. Prove that if $A_1, ..., A_n$ are independent events, then the same is true with any number of A_i replaced by their complement A_i^c. This really implies therefore that any selection of one set from each of $\sigma(A_1), \sigma(A_2), \ldots \sigma(A_n)$ is a set of mutually independent events.

11. Find an example such that A, B are independent and B, C are independent but $P(A \cup B|C) \neq P(A \cup B)$.

12. Prove that for any sequence of events A_n ,

$$P(\liminf A_n) \leq \liminf P(A_n)$$

13. Prove the *multiplication rule*. That if $A_1 \ldots A_n$ are arbitrary events,

$$P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2 A_1) \ldots P(A_n|A_1 A_2 \ldots A_{n-1})$$

14. Consider the unit interval with Lebesgue measure defined on the Borel subsets. For any point x in the interval, let $0.x_1 x_2 x_3 \ldots$ denote its decimal expansion (terminating wherever possible) and suppose A is the set of all points x such that $x_i \neq 5,\ i = 1, 2, \ldots$.

    (a) Prove that the set A is Borel measurable and find the measure of the set A.

    (b) Is the set A countable?

15. Give an example of a sequence of sets $A_n, n = 1, 2, ...$ such that $lim\ sup\ A_n = lim\ inf\ A_n$ but the sequence is not nested. Prove in this case that $P(lim\ sup\ A_n) = lim P(A_n)$.

16. In a given probability space, every pair of distinct events are independent so if $B \neq A$, then

$$P(A \cap B) = P(A)P(B)$$

    What values for the probabilities $P(A)$ are possible? Under what circumstances is it possible that

$$P(A \cap B) \leq P(A)P(B)$$

    for all $A \neq B$?

17. Prove that a $\lambda-$system does not need to be closed under general unions or finite intersections. For example let $\mathcal{F}$ consist of all subsets of $\{1, 2, 3, 4\}$ which have either 0 or 2, or 4 elements.

18. Suppose $\mathcal{F}_0$ is a Boolean algebra of sets and $A \in \sigma(\mathcal{F}_0)$ has the property that A is independent of every set in $\mathcal{F}_0$. Prove that $P(A) = 0$ or 1.

19. Prove: If $\mathcal{F}$ is both a π-system and a λ-system then it is a sigma-field.

20. Is the family consissting of all countable subsets of a space Ω and their complements a sigma-algebra?

21. Find $\limsup A_n$ and $\liminf A_n$ where $A_n = (\frac{1}{n}, \frac{2}{3} - \frac{1}{n}), n = 1, 3, 5, ...$ and $A_n = (\frac{1}{3} - \frac{1}{n}, 1 + \frac{1}{n}), n = 2, 4, 6, ...$

22. Consider a measure μ_0 defined on a Boolean algebra of sets $\mathcal{F}_0$ satisfying the conditions of Theorem 11. For simplicity assume that $\mu_0(\Omega) = 1$. Consider the class of sets $\mathcal{F}$ defined by

$$\mathcal{F} = \{A \subset \Omega; \mu^*(AE) + \mu^*(A^c E) = \mu^*(E) \text{ for all } E \subset \Omega\}.$$

Prove that $\mathcal{F}$ is a Boolean algebra.

23. Consider $\mathcal{F}$ as in Problem 22. Prove that if $A_1, A_2, ...$ disjoint subsets of $\mathcal{F}$ then $\mu^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu^*(A_i)$ so that this outer measure is countably additive.

24. Consider $\mathcal{F}$ as in Problem 22. Prove that $\mathcal{F}$ is a sigma-algebra.

25. Consider $\mathcal{F}$ as in Problem 22. Prove that if $A \in \mathcal{F}_0$ then $\mu^*(A) = \mu(A)$.

26. Prove or disprove: the family consisting of all finite subsets of a space Ω and their complements is a sigma-algebra.

27. Prove or disprove: the family consisting of all countable subsets of a space Ω and their complements is a sigma-algebra.

28. Find two sigma-algebras such that their union is not a sigma algebra.

29. Suppose P and Q are two probability measures both defined on the same sample space Ω and sigma algebra $\mathcal{F}$. Suppose that $P(A) = Q(A)$ for all events $A \in \mathcal{F}$ such that $P(A) \leq \frac{1}{2}$. Prove that $P(A) = Q(A)$ for all events A. Show by counterexample that this statement is not true if we replace the condition $P(A) \leq \frac{1}{2}$ by $P(A) < \frac{1}{2}$.