Chapter 1

Mathematical Prerequisites

1.1 Sets and sequences of Real Numbers

The real numbers $\Re$ form a *field*. This is a set together with operations of addition and multiplication and their inverse operations (subtraction and inverse). They are totally ordered in the sense that any two real numbers can be compared; i.e. for any $a, b \in \Re$, either $a < b,\ a = b,\ \ or\ \ a > b$. The set of real numbers, unlike the set of rational numbers, is uncountable. A set is *countable* if it can be put in one-one correspondence with the positive integers. It is is *at most countable* if it can be put in one-one correspondence with *a subset* of the positive integers (i.e. finite or countable). The set of rational numbers is countable, for example, but it is easy to show that the set of all real numbers is not. We will usually require the concept of "at most countable" in this course and often not distinguish between these two terminologies, i.e. refer to the set as countable. If we wish to emphasize that a set is infinite we may describe it as *countably infinite*.

A brief diversion: why do we need the machinery of measure theory? Consider the simple problem of identifying a uniform distribution on all subsets of the unit iterval $[0, 1]$ so that this extends the notion of length. Specifically can we define a "measure" or distribution P so that

1. $P(([a, b)) = b - a$ for all $0 \leq a \leq b \leq 1$
2. $P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$ for any disjoint sequence of sets $A_n \subset [0, 1], n = 1, 2, ...$
3. $P(A \oplus r) = P(A)$ for any $r \in [0, 1]$ where for $A \subset [0, 1]$,we define the shift of a set

$$A \oplus r = \{x \in [0, 1]; x - r \in A \text{ or } x - r + 1 \in A\}.$$

Theorem 1 *There is no function P defined on all the subsets of the unit interval which satisfies properties 1-3 above.*

The consequence of this theorem is that in order to define even simple continuous distributions we are unable to deal with *all* subsets of the unit interval or the real numbers but must restrict attention to a subclass of sets or events in what we call a "sigma-algebra".

The set of all integers is not a field because the operation of subtraction (inverse of addition) preserves the set, but the operation of division (inverse of multiplication) does not. However, the set of rational numbers, numbers of the form p/q for integer p and q, forms a field with a countable number of elements. Consider $A \subset \Re$. Then A has an upper bound b if $b \geq a$ for all $a \in A$. If b_0 is the smallest number with this property, we define b_0 to be the *least upper bound*. Similarly lower bounds and greatest lower bounds.

The real numbers is endowed with a concept of distance. More generally, a set $\mathcal{X}$ with such a concept defined on it is called a *metric space* if there is a function $d(x, y)$ defined for all $x, y \in \mathcal{X}$(called the distance between points x and y) satisfying the properties

1. $d(x, y) > 0$ for all $x \neq y$ and $d(x, x) = 0$ for all x.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Obviously the real line is a metric space with distance $d(x, y) = |x - y|$ but so is any subset of the real line. Indeed any subset of Euclidean space $\Re^n$ is a metric space. A metric space allows us to define the notion of neighbourhoods and open sets. In particular, a neighbourhood of a point x is a set of the form $\{y; d(x, y) < r\}$ for some radius $r > 0$. A subset B of a metric space is *open* if every point x in B has a neighbourhood entirely contained in B.Formally B is open if, for every $x \in B$, there exists $r > 0$ such that $\{y; d(x, y) < r\} \subset B$. Note that the whole metric space X is open, and trivially the empty set φ is open.

We say that a set E in a metric space has an open cover consisting of (possibly infinitely many) open sets $\{G_s, s \in S\}$ if $E \subset \cup_{s \in S} G_s$,or in other words if every point in E is in at least one of the open sets G_s. The set E is *compact* if every open cover has a finite subcover– i.e. if for any open cover there are finitely many sets, say $G_{s_i}, i = 1, ..., n$ such that $E \subset \cup_i G_{s_i}$. Compact sets in Euclidean space are easily identified- they are closed and bounded. In a general metric space, a compact set is always closed.

Now consider a sequence of elements of a metric space $\{x_n, \quad n = 1, 2, \ldots\}$. We say this sequence *converges* to a point x if, for all $\epsilon > 0$ there exists an $N < \infty$ such that $d(x_n, x) < \epsilon$ for all $n > N$. The property that a sequence converges and the value of the limit is a property only of the *tail* of the sequence- i.e. the values for n arbitrarily large. If the sequence consists of real numbers and if we define $l_N = sup\{x_n; n \geq N\}$ to be the least upper bound of the set $\{x_n; n \geq N\}$, then we know the limit x, provided it exists, is less than or equal to each l_N. Indeed since the sequence l_N is a decreasing sequence, bounded below, it must converge to some limit l, and we know that any limit is less than or equal to l as well. The limit $l = lim_{N\to\infty} l_N$ we denote commonly by $l = lim\ sup_{n\to\infty} x_n$.

It is easy to identify $l = lim\ sup$ of a sequence of numbers x_n by comparing it to an arbitrary real number a. In general, $l > a$ if and only if $x_n > a$ infinitely many times or infinitely often (i.e. for infinitely many subscripts n). Similarly $l \leq a$ if and only if $x_n > a + \epsilon$ at most finitely many times or finitely often for each $\epsilon > 0$.

We will deal throughout Stat 901 with subsets of the real numbers. For example, consider the set $\mathcal{O}$ of all *open intervals* $(a, b) = \{x; a < x < b\}$ and include $(a, a) = \phi$ the empty set. If we take the union of two (overlapping or non-overlapping) sets in $\mathcal{O}$ is the result in $\mathcal{O}$? What if we take the union of finitely many? Infinitely many? Repeat with intersections. These basic properties of open intervals are often used to describe more general *topologies* since they hold for more complicated spaces such as finite dimensional Euclidean spaces. Denote a closed interval $[a, b] = \{x; a \leq x \leq b\}$. Which of the above properties hold for closed intervals? Note that we can construct closed intervals from open ones provided we are permitted countably many operations of intersections for example:

$$[a, b] = \cap_{n=1}^{\infty} (a - 1/n,\ b + 1/n).$$

We shall normally use the following notation throughout this course. Ω is a fundamental measure (or probability, or sample) space. It is a set consisting of all *points* possible as the outcome to an experiment. For example what is the probability space if the experiment consists of choosing a random number from the interval $[0, 1]$? What if the experiment consists of tossing a coin repeatedly until we obtain exactly one head? We do not always assume that the space Ω has a topology (such as that induced by a metric) but in many cases it is convenient if the probability space does possess a metric topology. This is certainly the case if we are interested in the value of n random variables and so our space is $\Re^n$.

We denote by Ω a typical point in Ω. We wish to discuss events or classes of sets of possible outcomes.

Definition 2 *(Event) An* Event *A is a subset of Ω. The empty event ϕ and the whole space Ω are also considered events. However, the calculus of probability does not allow us in the most general case to accommodate the set of* all possible subsets of Ω *in general, and we need to restrict this class further.*

Definition 3 *(Topological Space) A topological Space $(\Omega, \mathcal{O})$ is a space Ω together with a class $\mathcal{O}$ of subsets of Ω. The members of the set $\mathcal{O}$ are called* open *sets. $\mathcal{O}$ has the property that unions of any number of the sets in $\mathcal{O}$ (finite or infinite, countable or uncountable) remain in $\mathcal{O}$, and intersections of finite numbers of sets in $\mathcal{O}$ also remain in $\mathcal{O}$. The* closed sets *are those whose complements are in $\mathcal{O}$.*

Definition 4 *(Some Notation)*

1. Union of sets $A \cup B$
2. Intersection of sets $A \cap B$
3. Complement : $A^c = \Omega \setminus A$
4. Set differences : $A \setminus B = A \cap B^c$.
5. Empty set : $\phi = \Omega^c$

Theorem 5 *(De Morgan's rules)* $(\cup_i A_i)^c = \cap_i A_i^c$ *and* $(\cap_i A_i)^c = \cup_i A_i^c$

Definition 6 *(Boolean Algebra) A Boolean Algebra (or algebra for short) is a family* $\mathcal{F}_{\prime}$ *of subsets of* Ω *such that*

1. $A, B \in \mathcal{F}_{\prime}$ *implies* $A \cup B \in \mathcal{F}_{\prime}$.
2. $A \in \mathcal{F}_{\prime}$ *implies* $A^c \in \mathcal{F}_{\prime}$.
3. $\phi \in \mathcal{F}_{\prime}$.

While Boolean algebras have satisfying mathematical properties, they are not sufficiently general to cover most probability spaces of interest. In particular, they may be used to model experiments with at most a finite number of possible outcomes. In the next chapter, we will deal with extending Boolean algebras to cover more general probability spaces.

1.2 Problems

1. Suppose we consider the space Ω of positive integers and define a measure by $P(A) = 0$ of the number of integers in A is finite, $P(A) = 1$ if the number is infinite. Does this measure satisfy the property of *countable additivity:*

$$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$$

for any disjoint sequence of sets $A_n \subset \Omega$, $n = 1, 2, ...$?

2. Prove that the equation $p^2 = 2$ is not satisfied by any rational number p.(Let $p = m/n$ where not both integers m, n are even).

3. The extended real number system consists of the usual real numbers $\{x; -\infty < x < \infty\}$ together with the symbols ∞ and $-\infty$. Which of the following have a meaning in the extended real number system and what is the meaning? Assume x is real $(-\infty < x < \infty)$.

   (a) $x + \infty$

(b) $x - \infty$

(c) $x(+\infty)$

(d) x/∞

(e) $\frac{x}{-\infty}$

(f) $\infty - \infty$

(g) ∞/∞

4. Prove: the set of rational numbers, numbers of the form p/q for integer p and q, has a countable number of elements.

5. Prove that the set of all real numbers is not countable.

6. Let the sets $E_n, n = 1, 2, ...$ each be countable. Prove that $\cup_{n=1}^{\infty} E_n$ is countable.

7. In a metric space, prove that for fixed x and $r > 0$, the set $\{y; d(x, y) < r\}$ is an open set.

8. In a metric space, prove that the union of any number of open sets is open, the intersection of a finite number of open sets is open, but the intersection of an infinite number of open sets might be closed.

9. Give an example of an open cover of the interval $(0, 1)$ which has no finite subcover.

10. Consider A to be the set of rational numbers $a \in Q$ such that $a^2 < 2$. Is there least upper bound, and a greatest lower bound, and are they in Q?

11. Show that any non-decreasing sequence of numbers that is bounded above converges.

12. Show that if $x \leq l_N$ for each $N < \infty$ and if l_N converges to some number l, then $x \leq l$.

13. Find an example of a double sequence $\{a_{ij}, \quad i = 1, 2, \ldots, \ j = 1, 2, \ldots\}$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

14. Define the set $\mathcal{O}$ of *open intervals* $(a, b) = \{x; -a < x < b\}$, $\infty \geq a \geq 0$, $\infty \geq b \geq 0$.

    (a) Verify that the union or intersection of finitely many sets in $\mathcal{O}$ is in $\mathcal{O}$.

    (b) Verify that the union of a countably infinite number of sets in $\mathcal{O}$ is in $\mathcal{O}$

(c) Show that the intersection of a countably infinite number of sets in $\mathcal{O}$ may not be in $\mathcal{O}$.

15. Prove the triangle inequality:

$$|a+b| \leq |a| + |b|$$

whenever $a, b \in \Re^n$.

16. Define the metric $d(X,Y) = \sqrt{E(X-Y)^2}$ on a space of random variables with finite variance. Prove the triangle inequality

$$d(X,Z) \leq d(X,Y) + d(Y,Z)$$

for arbitrary choice of random variables X, Y, Z.(Hint: recall that $cov(W_1, W_2) \leq \sqrt{var(W_1)}\sqrt{var(W_2)}$)

17. Verify that

$$[a,b) = \cap_{n=1}^{\infty}(a - 1/n,\ b).$$
$$(a,b) = \cup_{n=1}^{\infty}(a + 1/n,\ b - 1/n).$$
$$[a,b) = \cup_{n=1}^{\infty}[a,\ b - 1/n).$$

18. Let a_n be a sequence or real numbers converging to a. Prove that $|a_n|$ converges to $|a|$. Prove that for any function $f(x)$ continuous at the point a then $f(a_n) \to f(a)$.

19. Give an example of a convergent series $\sum p_n = 1$ with all $p_n \geq 0$ such that the expectation of the distribution does not converge; i.e. $\sum_n np_n = \infty$.

20. Define Ω to be the interval (0,1] and $\mathcal{F}_0$ to be the class of all sets of the form $(a_0, a_1] \cup (a_2, a_3] \cup ... \cup (a_{n-1}, a_n]$ where $0 \leq a_0 \leq ... \leq a_n \leq 1$. Then is $\mathcal{F}_0$ a Boolean algebra? Verify.

21. Prove that any open subset of $\Re$ is the union of countable many intervals of the form (a, b) where $a < b$.

22. Suppose the probability space $\Omega = \{1,2,3\}$ and $P(\varphi) = 0, P(\Omega) = 1$. What conditions are necessary for the values $x = P(\{1,2\}), y = P(\{2,3\}), z = P(\{1,3\})$ for the measure P to be countably additive?

23. Suppose a measure satisfies the property of *countable additivity:*

$$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$$

for any disjoint sequence of sets $A_n \subset \Omega$, $n = 1, 2, ...$?

Prove that for an arbitrary sequence of sets B_j,

$$P(B_1 \cup B_2 \cup ...) \leq P(B_1) + P(B_2) + ...$$

24. Prove for any probability measure and for an arbitrary sets $B_j, j = 1, 2, ..., n$

$$P(B_1 \cup B_2 \cup ...B_n) = \sum_{j=1}^{n} P(B_j) - \sum_{i<j} P(B_i B_j) + \sum_{i<j<k} P(B_i B_j B_k)....$$

25. Find two Boolean Algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ both defined on the space $\Omega = \{1, 2, 3\}$ such that the union $\mathcal{F}_0 \cup \mathcal{F}_1$ is NOT a Boolean Algebra.

26. For an arbitrary space Ω, is it true that

$$\mathcal{F}_0 = \{A \subset \Omega; A \text{ is a finite set}\}$$

is a Boolean algebra?

27. For two Boolean Algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ both defined on the space Ω is it true that the intersection $\mathcal{F}_0 \cap \mathcal{F}_1$ is a Boolean Algebra?

28. The smallest non-empty events belonging to a Boolean algebra are called the *atoms.* Find the atoms of

$$\mathcal{F}_0 = \{\varphi, \Omega, \{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{1, 4\}, \{2, 3, 4\}\}$$

where $\Omega = \{1, 2, 3, 4\}$.

29. The smallest non-empty events belonging to a Boolean algebra are called the *atoms.* Show that in general different atoms must be disjoint. If a Boolean algebra $\mathcal{F}_0$ has a total of n atoms how many elements are there in $\mathcal{F}_0$?