21. Mrs Jones made a rhubarb crumble pie. While she is away doing heart bypass surgery on the King of Tonga, her son William (graduate student in Stat-Finance) comes home and eats a random fraction X of the pie. Subsequently her daughter Wilhelmina (PhD student in Stat-Bio) returns and eats a random fraction Y of the remainder. When she comes home, she notices that more than half of the pie is gone. If one person eats more than a half of a rhubard-crumble pie, the results are a digestive catastrophe. What is the probability of such a catastrophe if X and Y are independent uniform on $[0,1]$?

Suppose that the fraction eaten by William is X and the fraction eaten by Wilhelmina of the remainder is Y. Then a catastrophe occurs if either $X > 1/2$ or if $(1-X)Y > 1/2$. Therefore we wish to compute

$$P[X > \frac{1}{2} \text{ or } X \leq \frac{1}{2} \text{ and } Y(1-X) > \frac{1}{2} | X + Y(1-X) > \frac{1}{2}]$$

where both X and Y are independent uniform$[0,1]$. Therefore

$$\begin{aligned} P[X &> \frac{1}{2}|X+Y(1-X) > \frac{1}{2}] + P[X \leq \frac{1}{2}, Y(1-X) > \frac{1}{2}|X+Y(1-X) > \frac{1}{2}] \\ &= \frac{\frac{1}{2} + \int_0^{1/2}\int_{1/(2(1-x))}^1 dydx}{\frac{1}{2} + \int_0^{1/2}\int_{(1/2-x)/(1-x)}^1 dydx} \\ &= \frac{\frac{1}{2} + \int_0^{1/2}(1 - \frac{1}{2(1-x)})dy}{\frac{1}{2} + \int_0^{1/2}\frac{1}{2(1-x)}dx} \\ &= \frac{2-\ln 2}{1+\ln 2} \end{aligned}$$

or about 0.77185.