Inequalities

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1 The Rearrangement Inequality

1. Four boxes contain \$10, \$20, \$50, \$100 bills respectively. From each box you may take 3, 4, 5, and 6 bills respectively. You have free choice of assigning the boxes to the numbers 3, 4, 5, 6. Take as much money as you can.

2. **Rearrangement Inequality.** Let $a_1 \le a2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be positive real numbers. Let π be any permutation on the set of positive integers less than or equal to n. Then
$$\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i b_{\pi(i)} \ge \sum_{i=1}^{n} a_i b_{n-i}\,.$$

   Proof. If $j < k$ and $b_{\pi(j)} > b_{\pi(k)}$, then by switching the values of $\pi(j)$ and $\pi(k)$, the change in the middle sum is
$$a_j b_{\pi(k)} + a_k b_{\pi(j)} - (a_j b_{\pi(j)} + a_k b_{\pi(k)}) = (a_k - a_j)\,(b_{\pi(j)} - b_{\pi(k)}) \ge 0\,.$$

3. **Extension of rearrangement inequality.** The usual statement of the rearrangement inequality uses two sequences. How can this be extended to three or more sequences? Let us extend the usual scalar product as follows. Let
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix} = a_1 b_1 c_1 + a_2 b_2 c_2 + \cdots + a_n b_n c_n$$
   and so on. Then when $0 \le a_1 \le \cdots \le a_n$, $0 \le b_1 \le \cdots \le b_n$, and $0 \le c_1 \le \cdots \le c_n$, we obtain
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix} \ge \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_{\pi(1)} & b_{\pi(2)} & \cdots & b_{\pi(n)} \\ c_{\rho(1)} & c_{\rho(2)} & \cdots & c_{\rho(n)} \end{bmatrix}$$
   for any two permutations π and ρ. Extensions to four or more sequences are obvious.

4. If we write
$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \ge \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}$$
   Now let $x_i = a_i^n$. We obtain the **AM-GM inequality**
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \cdots x_n}\,.$$

5. For any sequence $a_1, \cdots, a_n$ of positive numbers we have
$$n \le \frac{a_1}{a_{\pi(1)}} + \cdots + \frac{a_n}{a_{\pi(n)}}\,.$$

6. Nesbitt's inequality states that for positive a, b and c,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}\,.$$

Prove this using the rearrangement inequality.

7. We start with the inequality

$$\begin{bmatrix} 1 & 1 & 1 & a & b & c \\ 1 & 1 & 1 & a^{-1} & b^{-1} & c^{-1} \end{bmatrix} \leq \begin{bmatrix} 1 & 1 & 1 & a & b & c \\ a^{-1} & b^{-1} & c^{-1} & 1 & 1 & 1 \end{bmatrix}.$$

This gives us

$$6 \leq (a + a^{-1}) + (b + b^{-1}) + (c + c^{-1})\,.$$

Set $a = x_1/x_2$, $b = x_2/x_3$ and $c = x_3/x_1$, where $x_1, x_2, x_3 > 0$. Adding three to both sides we get

$$3^2 \leq (x_1 + x_2 + x_3)\,(x_1^{-1} + x_2^{-1} + x_3^{-1})\,.$$

We can write this in its more usual form

$$\frac{3}{x_1^{-1} + x_2^{-1} + x_3^{-1}} \leq \frac{x_1 + x_2 + x_3}{3}\,,$$

which is the **AM-HM inequality** for three variables. Clearly, it can be extended to n variables.

8. Use the rearrangement inequality to prove that

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x} \geq 1\,.$$

9. Use the AM-GM inequality to prove the **Cauchy-Schwarz inequality** namely that

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)\,(b_1^2 + b_2^2 + \cdots + b_n^2)\,.$$

10. Set

$$b_1 = b_2 = \cdots = b_n = 1$$

in the Cauchy-Schwarz inequality. Suppose $x_1 = a_1, \ldots, x_n = a_n$ are all nonnegative. Taking square roots we get we get the **AM-QM inequality**, namely that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2 + \cdots + x_n^2}{n}}$$

11. A surprising inequality known as **Carlson's inequality** can be proved using Cauchy-Schwarz.

$$\begin{aligned}(a_1 + \cdots + a_n)^2 &\le \left(a_1 \cdot 1 \cdot 1^{-1} + a_2 \cdot 2 \cdot 2^{-1} + \cdots + a_n \cdot n \cdot n^{-1}\right)^2 \\ &\le (1 \cdot a_1^2 + 2^2 \cdot a_2^2 + \cdots + n^2 \cdot a_n^2)\,(1 + 2^{-2} + 3^{-2} + \cdots + n^{-2})\,.\end{aligned}$$

We conclude that

$$(a_1 + a_2 + \cdots + a_n)^2 \le \frac{\pi^2}{6}\,(a_1^2 + 2^2 a_2^2 + \cdots n^2 a_n^2)\,,$$

which is Carlson's first inequality.

2 Sensible advice

- It is often a good idea to see where equality is obtained. For example, suppose we need to prove that $f(a,b,c) \ge 0$. If we discover that $f(a,b,c) = 0$ whenever $a = b$ and c is arbitrary then we know that we will not use AM-GM on all three variables. However, it is possible that the proof will involve AM-GM on a and b alone.

- Look for symmetries in the inequality. The proof may exploit that symmetry. For example with cyclic symmetry,
$$f(a,b,c) = f(b,c,a) = f(c,a,b)$$
the proof may involve writing down three inequalities and adding them up. Maybe!! If the inequalities is (completely) symmetric, we can impose any ordering assumption we like on the variables.

- Look for homogeneity in the variables. If $f(a,b,c) = f(\lambda a, \lambda b, \lambda c)$ for all nonzero λ, then you can impose an extra assumption such as $a + b + c = 1$, say, or $abc = 1$. It is like having more equations to use for free.

- Exploit convexity or concavity!

- The theory of inequalities is virtually complete. It is becoming increasingly difficult at the IMO to find original inequalities. The jury has been burnt in the past by posing well known inequalities as new problems. So the natural reaction is to move towards problems requiring grubby unusual methods. **You must be prepared to get your hands dirty.**

3 More problems

1. For nonnegative a, b, c we have

$$(a+b)\,(b+c)\,(c+a) \ge 8\,abc\,.$$

2. For positive values

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \ge 2\,.$$

3. For $0 \le a, b, c, \le 1$

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (a-1)(b-1)(c-1) \le 1\,1.$$

4. Let

$$a = (m^{m+1} + n^{n+1})/(m^n + n^n)$$

where m and n are positive integers. Prove that

$$a^m + a^n \ge m^m + n^n\,.$$

5. For nonnegative variables

$$x(x-z)^2 + y(y-z)^2 \ge (x-z)(y-z)(x+y-z)\,.$$

6. For positive variables

$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \le \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}\,.$$

7. Suppose variables are nonnegative, and that $ab + bc + cd + da = 1$. Show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \ge \frac{1}{3}\,.$$

8. Let P be a polynomial with positive coefficients. Prove that if

$$P(x^{-1}) \ge [P(x)]^{-1}$$

holds for $x = 1$, then it holds for every $x > 0$.

9. The Fibonacci sequence is $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2^i} < 2\,.$$

10. Let $0 < a \le b \le c \le d$. Then $a^b b^c c^d d^a \ge b^a c^b d^c a^d$.

11. For all x and y,

$$-\frac{1}{2} \le \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \le \frac{1}{2}\,.$$

12. For nonnegative variables

$$\frac{(a+b)^2}{2} + \frac{a+b}{4} \ge a\sqrt{b} + b\sqrt{a}\,.$$

13. The positive numbers $a_1, \ldots, a_n$ and $b_1 \le \cdots \le b_n$ satisfy

$$a_1 \le b_1$$

$$a_1 + a_2 \le b_1 + b_2$$

and so on to

$$a_1 + \cdots + a_n \le b_1 + \cdots + b_n\,.$$

Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \le \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}\,.$$