

Games and Strategies

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1 Two person games of perfect information

We start by considering games of perfect information where two players take turns moving. We exclude games such as chinese checkers, involving more than two players.

- In many such games, it is possible in principle to enumerate all possible situations that can arise in the game and thereby define a set P of possible “positions.”
- The game is said to be of perfect information whenever it is the case that the present position of the game is always completely known to both players. In games of imperfect information—most card games—there is a position that the game is in. However, the players have incomplete, and usually different, information about the position of the game.
- We are given a set of rules M for how to move from one position to another. The set M can be thought of as a collection of ordered pairs:

$$M = \{ (x, y) : x, y \in P \}.$$

The statement $(x, y) \in M$ denotes the possibility that if the game is in position x then there is a legal move which puts the game in position y . In some games the legal moves of one player are distinct from the legal moves of the other player. (Chess and checkers are such games because the players move different pieces. To begin, we shall consider two person games where the set of legal moves are the same in any given situation for both players.) If one player is in position x and legal moves to y , then it is the other player’s turn.

- We are also given a set of losing positions—a subset of P . A losing position is one such that if either player finds himself in that position and it is his turn to move, then he loses. In many games, this occurs because there is no legal move. For example all counters in a game have been removed from the board, and a legal move requires the removal of a counter.

Assuming that ties are impossible, strategical analysis of the games divides P into two subsets W and L where

$$P = W \cup L \quad \text{and} \quad W \cap L = \emptyset,$$

such that with best play on both sides, any player required to make a legal move and having a position from W will win, and any player required to make a legal move and having a position in L will lose. A *winning strategy* then defines a function $S : W \rightarrow L$ such that $(x, S(x)) \in M$ for every $x \in W$.

1.1 Example

There are 100 counters on a table. On his turn, a player may remove a single counter, or a prime number of counters. The person who takes the last counter wins. In this case

$$P = \{0, 1, 2, \dots, 100\}$$

with starting position $x = 100$. A move is an ordered pair $(x, y) \in P \times P$ where

$$x - y \in \{1, 2, 3, 5, 7, 11, \dots\}.$$

In this game, it turns out that most of the prime numbers are red herrings. The first number missing from the list above is 4. Therefore, if you have 4 you must lose. Let $L = \{4n : 0 \leq n \leq 25\}$ and let $W = P - L$. For any $x = 4n + m \in W$, where $1 \leq m \leq 3$, we take m counters from the table. So $S(4n + m) = 4n$ is a winning strategy for the player with a position in W .

1.2 Sundry Problems

In the following games between Alice and Bob, Alice always goes first.

1. Alice and Bob play the game in the example above, except that instead of removing a number number or one counter, at each turn any feasible power of 2 can be removed. Find L and W and the winning strategy S . Who wins?
2. Same as above, except that there are 10^7 counters on the table(!) At any turn, the p^n counters can be removed, where p is any prime and n is any nonnegative integer. Find L , W and the winning strategy for Alice or Bob.
3. A regular 2008-gon is given. Alice and Bob take turns drawing diagonals within the 2008-gon by connecting pairs of distinct vertices by line segments. There is a restriction that no diagonal may intersect an earlier one in the interior of the 2008-gon. Who wins?
4. Alice and Bob start with 1. Alice multiplies 1 by any number from 2 to 9. Bob then multiplies the result by any number from 2 to 9, and so on. The winner is the first person to get one million or more.
5. Alice crosses out any 2^7 of the numbers 0, 1, ..., 255, 256. Then Bob crosses out any 2^6 numbers, and so on, until Bob finally crosses out $2^0 = 1$ number. Since $2^7 + 2^6 + \dots + 2 + 1 = 2^8 - 1$, there must be two numbers left, say a and b . Bob pays, $|a - b|$ to Alice. What are the optimal strategies for Alice and Bob if Alice wants to get as much money as possible and Bob wishes to lose as little as possible.
6. Alice and Bob take turns placing a "+" sign or a "-" sign in front of one of the numbers in the sequence 1 2 3 ... 19 20. After all 20 signs have been placed, Bob wins the *absolute value* of the sum from Alice. Find the best strategy for each player, and determine how much Bob wins under best play by both sides.
7. Alice and Bob alternately write positive integers $\leq p$ on the blackboard. It is illegal in the game to write any integer on the board which is a divisor of an integer previously written. The loser is the person who cannot write a number. For each positive integer p determine who wins with best play by both sides.

- Alice places a knight on a square of an 8×8 board. Then Bob makes a legal knight's move. Then Alice makes a legal move, but is not allowed to place the knight on any square visited before, etc. The loser is the person who cannot move. Who wins under best play by both sides?
- On a piece of paper is a row of n empty boxes. Alice and Bob take turns, each writing an "S" or an "O" into a previously blank box. The winner is the one who completes an "SOS" in consecutive boxes. For which n does Bob have a winning strategy?

2 Two person games involving uncertainty and chance

- Alice gets two pieces of paper and writes an integer (positive or negative) on each piece. The two integers are required to be different. She then hides the pieces of paper in each hand, and lets Bob choose a hand. The integer in the chosen hand is revealed to Bob. Bob must then guess which of the two numbers is bigger—the one revealed or the one still hidden. If he guesses correctly he wins \$1. If he is wrong he loses \$1. Is there any strategy Bob can use to guarantee that he has a better than even chance of winning?
- Alice shuffles a deck of cards. She then deals the cards face up one at a time from the top of the deck. At any moment, Bob is permitted to stop Alice and bet \$1 that the next card will be red. He bets once and only once. If he chooses never to stop Alice, then the final card of the deck is the basis for the bet. Obviously, Bob can break even by betting on the very first card (or the k^{th} card for that matter for fixed k). Is there a strategy that is better than this?
- In the cruel dictatorship of Ottawania, the dissolute leaders like to play party games using hats and the downtrodden commoners. One of their favourite games is the following. Ten people are lined up in a row so that they are all facing in the same direction. A hat is placed on each person's head, so that the i^{th} person in line can only see the hat of the first $i - 1$ people in front of him. Each hat is either red or blue. Starting from the back of the line—that is the tenth person who can see nine people in front of him—each person must guess the colour of his hat. The i^{th} person in line can hear the colours that are called out behind him, but can only see the colours of the people in front of him and cannot see his own colour. At the end of the game, the people who guessed incorrectly are taken out and executed.

By guessing, each the "contestants" can save five of ten on average. How much better than this can they do, assuming they can collaborate on a strategy beforehand, but cannot otherwise communicate during the game?

- Another popular game in Ottawania also uses red and blue hats. Three contestants sit together, each with a blue hat or a red hat. The choice of colour of each hat is determined by a fair coin toss (but the result of each toss is not communicated to the contestants. However, each can see the hat of the other two but cannot see his own. Simultaneously each player must announce the colour of his own hat. Each contestant can pass, if he wishes to, by remaining silent. Unlike the last game, the three contestants form a team. The team wins if every person who chooses not to pass is correct. The team wins if everyone chooses to pass, or if at least one person guesses incorrectly. Once again, the three can collaborate on a strategy beforehand, but cannot otherwise communicate during the game. Winning teams are honoured by a year's

labour in the fields. Losing teams are all executed. Devise a strategy to maximise the chance that the team will win.

5. A little known fact about Ottawania is that there are infinitely many commoners—denumerably many to be precise. A popular—if rarely staged—game in Ottawania using red and blue hats is the following. Contestants, numbered 1, 2, 3, ..., are given red and blue hats. At a precise moment, all contestants are revealed to each other so that each gets to see the hat colours of all (infinitely many) others. No communication is allowed. Each contestant is taken aside and asked to guess the colour of his own hat. Contestants can agree on a strategy beforehand, but cannot otherwise communicate during the game. All contestants will be executed unless *only finitely many* guess incorrectly. Is there a strategy that will guarantee they will all survive?
6. This game is identical to the previous game except that the rule for execution is different. Contestants will be executed unless that all guess correctly or they all guess incorrectly. Is there a strategy in this case?