

PUTNAM TRAINING PROBLEMS 2001.5

Period pieces

1. Let $f : \mathbb{R} \rightarrow \mathbb{R} - \{3\}$ be a function with the property that for some $\omega > 0$

$$f(x + \omega) = \frac{f(x) - 5}{f(x) - 3},$$

for all real x . Prove that f is periodic.

2. If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *not* injective and there exists a function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x + y) = g(f(x), y)$ for all $x, y \in \mathbb{R}$, show that f is periodic.

3. Find all real values α with the property that a sequence satisfying the recurrence relation $x_{n+1} + x_{n-1} = \alpha x_n$, $n \in \mathbb{N}$, is periodic.

4. Let b_n be the last digit of the number

$$1^1 + 2^2 + 3^3 + \dots + n^n.$$

Prove that the sequence b_1, b_2, b_3, \dots is periodic with period 100.

5. (Putnam 1955 B5.) Given an infinite sequence of 0's and 1's and a fixed integer k , suppose there are no more than k distinct blocks of k consecutive terms. Show that the sequence is eventually periodic.
6. (Putnam 1964 A4.) Let p_n , $n = 1, 2, \dots$, be a bounded sequence of integers which satisfies the recursion

$$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}}.$$

Show that the sequence eventually becomes periodic.

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function such that the set $\{f(n) : n \in \mathbb{N}\}$ has infinitely many elements. Prove that the period of f is irrational.
8. The positive integers a_1, a_2, a_3, \dots , all less than or equal to 2001, satisfy the following condition: $a_m + a_n$ is divisible by a_{m+n} for all positive integers m, n . Prove that this sequence is eventually periodic.