SPECIAL K
Saturday, October 28, 2000
9am-12 noon

1. Let $x > 1$ be a real number, and $n > 1$ be an integer. Prove that

$$\sqrt[n]{x} < 1 + \frac{x-1}{n}.$$

2. Find the smallest (by area) right-angled triangle with integral sides in which a square with integer sides can be inscribed so that an angle of the square coincides with the right angle of the triangle.

3. Let S be a set of points in the plane. A circle C is said to be *framed* by S if C has a diameter whose endpoints both lie in S. Find all sets S of four points in the plane such that, for any two circles C_1 and C_2 framed by S, the set $S \cap C_1 \cap C_2$ is nonempty.

4. Let f be a real-valued continuous function of a real variable with the property that

$$\lim_{x \to +\infty} f(f(x)) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(f(x)) = -\infty$$

Prove that $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist and are infinite.

5. Peter tells Ian and Christopher that x and y are two integers with $1 < x < y$ and $x + y \leq 30$. Peter then gives Christopher the value of $x + y$ and Ian the value of xy.

(1) Ian says: “I don’t know the values of x and y.”
(2) Christopher replies: “I knew that you didn’t know the values of x and y .”
(3) Ian responds: “Oh, then I do know the values of x and y.”
(4) Christopher exclaims: “Oh, then so do I.”

Prove that either Ian has made a mistake or Christopher has made a mistake.