The Bernoulli Trials 2001

by

Ian VanderBurgh and Christopher G. Small

It was a great turnout on the first Saturday in March 2001, as 37 contestants matched wits for the coveted title of Bernoulli Trials Champion. As always, the last person left standing in any Bernoulli Trials competition owes this status to a combination of mathematical prowess and old fashioned good luck.

Contestants in the Bernoulli Trials proceed through the trials answering each mathematical question by "true" or "false". A contestant can carry one mistake into future rounds, but after two mistakes he or she is eliminated from formal competition. As no solutions need be given, the contestant who is truly clueless can flip a mental coin and hope it lands the right way. (No calculators are allowed, but randomisers such as coins are considered within contest regulations.)

After 11 rounds of competition there were two battle-weary mathematicians left. Sabin Cautis, a veteran of many IMO and Putnam contests had accumulated one error on a previous round. His adversary, Marshall Drew-Brook, was a relative newcomer who had shown outstanding tenacity in solving one question after another. Like Cautis, he had also accumulated one error. So it came down to a final question on the 12th round. If both answered the same way, another question would be required. If their answers were different, a winner would be declared. As fate would have it, the answers were different.

First place in the competition goes to Marshall Drew-Brook, who won 200 dollars (awarded in coins). Sabin Cautis took second place and received 100 dollars. Cash prizes were also awarded to people ranked 3rd through 5th in the Bernoulli Trials. Third place went to Joel Kamnitzer, fourth place to Mark Mann, and fifth place to Masoud Kamgarpour.

The Questions:

1. To win a contest, a contestant must answer questions in order from a list of 2001 questions.

   - Questions are chosen at random from the list, and continue until he gives an incorrect answer. Of course, no question is ever asked twice.
   - It is known that 501 of the questions are too difficult for him. So he will answer these questions incorrectly if any is asked. He will answer the remaining questions correctly.

   TRUE or FALSE? The probability that he misses on the 501st question is

$$\frac{\binom{1500}{1000}}{\binom{2001}{1500}}.$$

2. Tony and Maria are training for a race by running all the way up and down a 700 meter long slope.

   - They run up the slope at constant speeds, but their speeds differ.
   - Coming down the slope, each runs at double his or her uphill speed.
   - Maria reaches the top first, and immediately starts running back down, meeting Tony 70 m from the top.

   TRUE or FALSE? When Maria reaches the bottom of the hill, Tony is 240 metres behind.

3. Altitude AD of triangle △ABC is a diameter of the circle shown. △ABC is equilateral. The circle intersects AB and AC at E and F respectively.

   TRUE or FALSE?

$$\frac{\text{EF}}{\text{BC}} = \frac{3}{4}.$$

4. Define
$$H_n = \frac{1}{n} + \frac{1}{n+3} + \ldots + \frac{1}{n+3(n-1)}.$$
TRUE or FALSE?
$$\lim_{n\to\infty} H_n \geq \frac{1}{2}.$$

5. Let $d(m)$ denote the number of divisors of m including 1 and m itself.

   TRUE or FALSE? There exist infinitely many positive integers k such that
$$d\left(2001^n\right) = 2\,k\,n + 1$$
   has a solution in positive integer n.

6. Alice and Barbara play the following game.

   - Alice and Barbara alternate choosing numbers from 1 to 2001, inclusive.
   - A number, once chosen cannot be chosen again.
   - Play continues until exactly two numbers are left.
   - Alice wins if the remaining numbers are relatively prime. Barbara wins if the two numbers are not relatively prime.
   - Alice goes first.

   TRUE or FALSE? With best play by both sides, Barbara wins.

7. A tetrahedron and an octahedron are built from a common stock of equilateral triangles. The volume of the tetrahedron is 1.

   TRUE or FALSE?

   The volume of the octahedron is 4.

8. Let f be a positive continuous real-valued function defined on the positive real numbers, satisfying
$$\frac{1}{a-b}\int_b^a f(x)\,dx = \sqrt{f(a)\,f(b)}$$
for all $a > b > 0$. Suppose also that $f(1) = 2001$.

TRUE or FALSE?
$$f(x) = 2001$$
for all $x > 0$.

9. In the following equation, each letter stands for a numeral in base 10:
$$6 \times \text{FORWAT} = 7 \times \text{WATFOR}$$

TRUE or FALSE? $W = 4$.

10. TRUE or FALSE?

There exists a triangle ABC with the tangents of the internal angles satisfying
$$\tan A = x,$$
$$\tan B = 1 + x,$$
$$\tan C = 1 - x$$
for some real value x.

11. Suppose that
$$x^{x^{2001}} = 2001,$$
where $x > 0$.

TRUE or FALSE?
$$x > 12.5^{0.0014}.$$

12. TRUE or FALSE?
$$2^{1001} \,\Big|\, \lfloor (1+\sqrt{3})^{2001} \rfloor.$$

Hints:

1. To miss on the 501st question, the first 500 questions must have been selected from the 1500 that were not too hard, and the 501st from the 501 that were too hard.

2. Consider Tony and Maria as each running 2100 m at constant speeds (the first 1400 m being the "uphill" portion).

3. $$\frac{\text{EF}}{\text{BC}} = \frac{\text{AE}}{\text{AB}} = \frac{\text{AE/AD}}{\text{AB/AD}} = \frac{\cos 30}{\sec 30}.$$

4. Note that
$$H_n = \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)$$
where $f(x) = 1/(1+3x)$. Show that $\lim H_n = (\ln 4)/3$. No calculators are allowed for the last step!

5. Since $2001 = 3 \times 23 \times 29$, it follows that $d(2001^n) = (n+1)^3$. So $(n+1)^3 = 2\,k\,n + 1$ reduces to
$$n^2 + 3\,n + (3 - 2k) = 0.$$
Thus
$$n = \frac{-3 + \sqrt{8\,k - 3}}{2}$$
which will be a positive integer provided $8k - 3$ is a perfect square.

6. Divide the numbers into pairs as
$$\{2n - 1,\, 2n\}$$
for $n = 1, \ldots, 1000$. This leaves 2001 by itself. Suppose Alice chooses 2001 as her starting number. For each number that Barbara chooses, Alice can choose the other number of the pair.

7. Let the side length of each of the solids be s. Notice that the octahedron is the combination two square based pyramids with all side lengths s. Therefore, the volume of the octahedron is
$$2 \times \frac{1}{3} \times (\text{base of pyramid}) \times (\text{height of pyramid}).$$
Write this as a function of s. Show that the volume of the tetrahedron is
$$\frac{\sqrt{2}}{12} s^3$$
which is one, by assumption.

8. Consider

$$f(x) = \frac{2001}{x^2}.$$

9. Let $x = \text{FOR}$ and let $y = \text{WAT}$. Then

$$6\,(1000\,x + y) = 7\,(1000\,y + x).$$

This reduces to

$$5993\,x = 6994\,y.$$

But 5993 and 6994 are both divisible by 13. So $461\,x = 538\,y$. However, 461 and 538 are relatively prime!

10. For the angles of a triangle,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

So if x, $1+x$ and $1-x$ were the tangents, then we would have

$$\begin{aligned} x + 1 + x + 1 - x &= x(1-x)(1+x) \\ x + 2 &= x - x^3 \\ x^3 &= -2 \\ x &= -\sqrt[3]{2} \end{aligned}$$

Thus $x < 0$ and $1 + x < 0$. Is this possible?

11. This would be trivial with an illegal calculator. However, ten minutes is enough time to number crunch this, even without a calculator. To do so is to miss the fact that there is a more mathematically interesting argument.

Obviously there are no solutions for $0 < x \leq 1$. Since

$$f(x) = x^{x^{2001}}$$

is increasing for $x > 1$, there is at most one solution for $x > 1$. It is easy to check that

$$x = 2001^{1/2001}$$

is a solution. But

$$2001^{1/2001} = \left(\sqrt[3]{2001}\right)^{1/667} > 12.5^{1/667} > 12.5^{0.0014}.$$

12. We can write

$$\begin{aligned} \lfloor (1+\sqrt{3})^{2001}) \rfloor &= (1+\sqrt{3})^{2001} + (1-\sqrt{3})^{2001} \\ &= (1+\sqrt{3})(4+2\sqrt{3})^{1000} + (1-\sqrt{3})(4-2\sqrt{3})^{1000} \\ &= 2^{1000}[(1+\sqrt{3})(2+\sqrt{3})^{1000} + (1-\sqrt{3})(2-\sqrt{3})^{1000})]. \end{aligned}$$

Can an additional factor of 2 be pulled out?

Answers:

1. True.
2. False.
3. True.
4. False.
5. False.
6. False.
7. True.
8. False.
9. True.
10. False.
11. True.
12. True.