The Bernoulli Trials 2000

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Since 1997, the Bernoulli Trials, an undergraduate mathematics competition, has been held at the University of Waterloo. This is a double knockout competition. At the start of each round, students are presented with a mathematics statement which can be true or false. They have 10 minutes to determine the truth or falsehood of the proposition, and drop out after their second incorrect answer.

In 1999, there were 36 student participants. The competition lasted for 3.5 hours and 13 rounds, after which the first four places where clearly determined. The winner was **Scott Sitar**, who was the sole contestant not to be eliminated at the end of 12 rounds. Second place went to **Megan Davis**, who won a 13^{th} round tie-breaker with 3^{rd} placed **Dennis The**. **Adrian Tang** came in 4^{th}, having survived to round 11. In keeping with the nature of the answers required, the prizes supplied by the Dean of Mathematics were awarded in coins: 200 dollars (100 “toonies”) for first, 100 dollars (“loonies”) for second, 70 dollars for third, and 30 dollars for fourth.

1. A deck of 2000 cards has the numbers from 1 to 2000 labelled consecutively in order from top to bottom. The deck is shuffled as follows. The second card from the top is placed on the top card, the third card is placed below these two, the fourth above these three, the fifth below these four, and so on, until the 2000th card is placed above the remaining 1999.

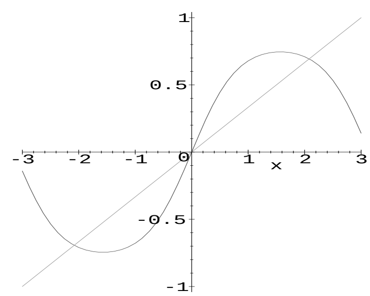
   TRUE or FALSE? At the completion of this shuffle, every card is in a different position in the deck than where it started. That is, for every $i = 1, \ldots, 2000$ the card labelled i is not in position i.

   TRUE. If a card is in position i, then it is moved to position
$$\frac{2001+i}{2}$$
   if i is odd, and position
$$\frac{2002-i}{2}$$
   if i is even. So no card will remain fixed.

2. **TRUE OR FALSE?**

   The equation
$$\sin(\sin(\sin(x))) = x/3$$
   has exactly one solution in real values x.

   FALSE. There are exactly 3 solutions. Note that all solutions must lie in the interval $[-3, +3]$. Let
$$f(x) = \sin(\sin(\sin(x)))$$
   To prove that there are 3 solutions it suffices to show that
$$f'(0) > 1/3$$



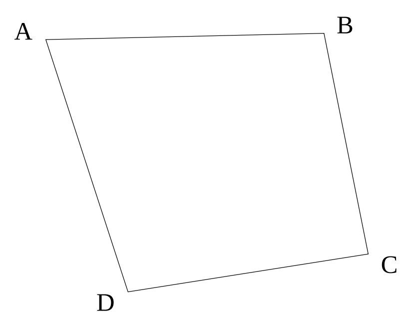
$$f(3) < 1$$

and

$$f(-3) > -1$$

and that f is concave on $(0,3]$ and convex on $[-3,0)$.

3. Let ABCD be a planar convex quadrilateral labelled clockwise as shown:



Suppose that

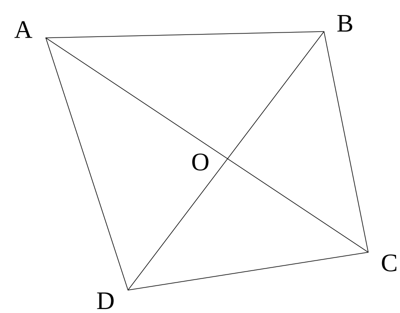
$$(ABC) \leq (BCD) \leq (CDA) \leq (DAB)$$

where (RST) represents the area of triangle RST.

TRUE or FALSE? AD is parallel to BC.

(A quadrilateral is said to be *convex* if no vertex is within the triangle formed by the other three vertices.)

TRUE. Let O be the intersection of diagonals AD and BC. Then



$$(ABC) = (AOB) + (BOC)$$

$$(BCD) = (BOC) + (COD)$$

Together these give $(AOB) \leq (COD)$.

Similarly

$$(CDA) = (COD) + (AOD)$$

$$(DAB) = (AOD) + (AOB)$$

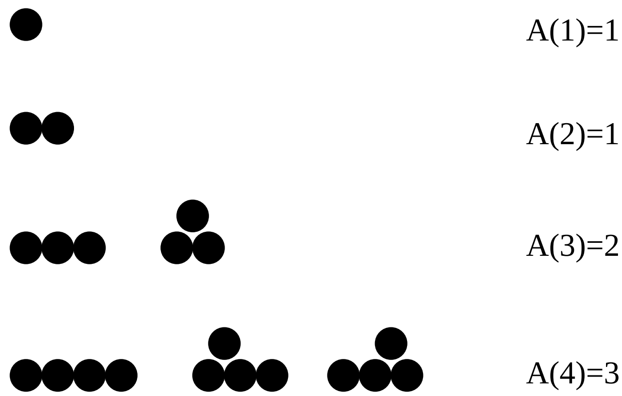
which gives us $(COD) \leq (AOB)$. So

$$(COD) = (AOB)$$

implying that $(ABC) = (DBC)$. This implies that AD is parallel to BC.

4. We arrange dimes in rows on top of each other according to the following rules:

   - each coin must touch the next in its row;
   - each coin except those in the bottom row touches two coins on the row below.



Let $A(n)$ be the number of distinct ways to arrange n coins. For example, $A(4) = 3$ as shown.

TRUE or FALSE? $A(n)$ is the n-th Fibonacci number: $A(1) = 1$, $A(2) = 1, \ldots$

$$A(n+2) = A(n) + A(n+1)$$

FALSE. The values $A(1), A(2), \ldots, A(6)$ are their respective Fibonacci numbers, but thereafter, the pattern breaks down.

5. **TRUE or FALSE?** For every integer $n \geq 3$, the equation

$$x^n + y^n = z^{n+1}$$

has infinitely many solutions in positive integers x, y and z.

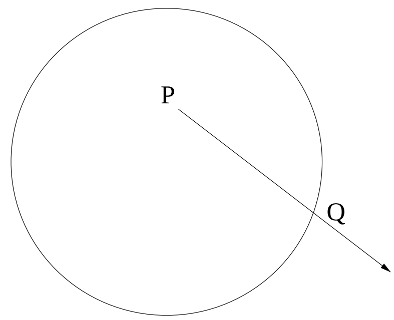
TRUE. Given integers a and b, define

$$z = a^n + b^n$$

$$x = a\,z$$

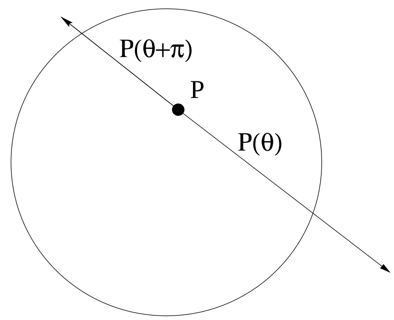
$$y = b\,z$$

6. Consider a point P at random inside a circle of diameter 2. From P, a ray is drawn in a random direction, and intersects the circumference of the circle at Q.



TRUE OR FALSE? The average length of PQ is 1.

FALSE. Let $P(\theta)$ be the length of the ray from point P in the direction θ to the point Q on the boundary. Now



$$P(\theta) + P(\theta + \pi) < 2$$

with probability one. So, letting $\mathcal{E}$ denote the expectation operator,

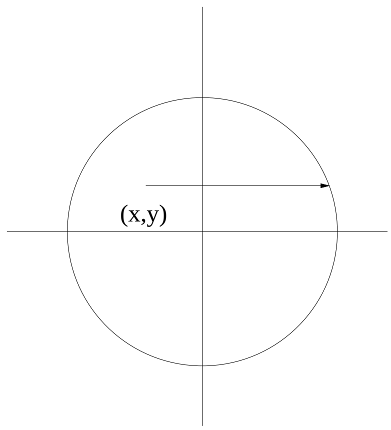
$$\mathcal{E}\, P(\theta) + \mathcal{E} P(\theta + \pi) < 2$$

Since

$$\mathcal{E}\, P(\theta) = \mathcal{E}\, P(\theta + \pi)$$

it follows that $\mathcal{E}\, P(\theta) < 1$.

The expectation can be computed exactly. (The following argument is due to Ken Davidson.) As the circle is rotationally symmetric, we can assign a direction to the ray, say in the positive x direction. We standardize the circle to be the unit circle in the x,y-plane.



The expected length is

$$\int_{-1}^{+1} \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \left(\sqrt{1-y^2} - x\right) \frac{dx\, dy}{\pi} = \int_{-1}^{+1} [2(1-y^2) - 0] \frac{dy}{\pi} = \frac{8}{3\,\pi} < 1$$

7. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be any n vectors in $\mathbf{R}^n$ such that $||\mathbf{v}_i|| = 1$, for all $i = 1, \ldots, n$.

TRUE OR FALSE? It is always possible to select $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$, so that

$$||\epsilon_1 \mathbf{v}_1 + \cdots + \epsilon_n \mathbf{v}_n|| \geq \sqrt{n}$$

and $\delta_1, \ldots, \delta_n \in \{-1, +1\}$, so that

$$||\delta \mathbf{v}_1 + \cdots + \delta_n \mathbf{v}_n|| \leq \sqrt{n}$$

TRUE. Suppose that $\epsilon_1, \ldots, \epsilon_n$ are chosen *randomly* with each probability for any selection. Then the expected value of the square of the L.H.S. is

$$\begin{aligned} \mathcal{E}\left(||\epsilon_1 \mathbf{v}_1 + \cdots + \epsilon_n \mathbf{v}_n||^2\right) &= \sum_{i=1}^{n} \mathcal{E}(\epsilon_i^2)\, ||\mathbf{v}_i||^2 \\ &= \sum_{i=1}^{n} ||\mathbf{v}_i||^2 \\ &= n \end{aligned}$$

So choices of $\epsilon_1, \ldots, \epsilon_n$ and $\delta_1, \ldots, \delta_n$ must be possible.

8. **TRUE or FALSE?** For all $0 < x < 1$,

$$\frac{d^{2000}}{dx^{2000}} [\ln(x) \ \ln(1-x)] < 0$$

TRUE.

$$\begin{aligned}\frac{d}{dx} [\ln(x) \ \ln(1-x)] &= \frac{\ln(1-x)}{x} - \frac{\ln(x)}{1-x} \\ &= \sum_{m=1}^{\infty} \left[\frac{(1-x)^{m-1}}{m} - \frac{x^{m-1}}{m} \right]\end{aligned}$$

The 1999th derivative of each term in the sum is negative or zero, and there is a term which is negative.

9. Two players, Arthur and Barbara, take turns selecting numbers from the set

$$\{1, 2, 3, 4, \ldots, 9\}$$

A number, after selection, cannot be selected in a subsequent round. The first player to obtain a set of 3 numbers totalling 15 is the winner.

TRUE or FALSE? With best play by both sides, the first player (Arthur) can force a win.

FALSE. Arrange the numbers in a magic square:

$$\begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix}$$

The only triples that add to 15 are rows columns and diagonals. This is tic-tac-toe, which is a draw with best play by both sides.

10. Let $f : I\!R \to I\!R$ be a continuous function. Suppose that for every rational number q there exists a positive integer N such that $f^n(q) = 0$ for all $n \geq N$, where f^n denotes the n-fold iteration of f.

TRUE or FALSE? For every real number t

$$\lim_{n\to\infty} f^n(t) = 0$$

FALSE. Let $f(x) = \pi - 2\,|\pi - x|$ when $|x - \pi| < \pi/2$ and $f(x) = 0$ when $|x - \pi| \geq \pi/2$. This function satisfies the conditions of the statement. However $f(\pi) = \pi$. Therefore the conclusion fails.

11. **TRUE or FALSE?**

$$\sum_{k=0}^{\infty} \frac{8k^3 + 4k + 1}{(2k)!} < \sum_{k=0}^{\infty} \frac{(2k+1)^3 + 4k + 3}{(2k+1)!}$$

FALSE. In fact they are equal. We can check the value of

$$\sum_{k=0}^{\infty} (-1)^k \frac{k^3 + 2k + 1}{k!}$$

which is the LHS - RHS. We have

$$\sum_{k=0}^{\infty} (-1)^k \frac{k(k-1)(k-2)}{k!} = -e^{-1}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{k(k-1)}{k!} = e^{-1}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{k}{k!} = -e^{-1}$$

Let

$$A_n = \sum_{k=0}^{\infty}(-1)^k \frac{k^n}{k!}$$

Then the equations become

$$A_3 - 3A_2 + 2A_1 = -e^{-1}$$

$$A_2 - A_1 = e^{-1}$$

$$A_1 = -e^{-1}$$

Solving for each A_n we get $A_2 = 0$ and $A_3 = e^{-1}$. So

$$\sum_{k=0}^{\infty}(-1)^k \frac{k^3 + 2k + 1}{k!} = A_3 + 2A_1 + e^{-1} = 0$$

12. Consider a sequence of positive integers

$$a_0,\ a_1,\ a_2,\ \ldots$$

with the property that a_n equals the number of positive divisors of a_{n-1}. (The number a_i has both 1 and a_i as divisors.) We set $a_0 = 2000!$.

TRUE or FALSE? For some positive integer n the number a_n is a perfect square.

TRUE. The trick in this problem is to think about the end of the sequence rather than the beginning. It turns out that there will be a perfect square if and only if a_0 is composite.

Note that the sequence decreases to 2, after which it stays constant. Let i be the first index such that $a_i = 2$. Then a_{i-1} is an odd prime, and a_{i-2} is a perfect square.

13. **TRUE or FALSE?** There exists a function

$$f : [-1, +1] \to \mathbb{R}$$

with continuous second derivative such that

$$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$$

converges and

$$\sum_{n=1}^{\infty} |f\left(\frac{1}{n}\right)|$$

diverges.

FALSE. Suppose f exists. Then there exists c_n such that $0 < c_n < 1/n$ and

$$f(n^{-1}) = f(0) + \frac{f'(0)}{n} + \frac{f''(c_n)}{2n^2}$$

Since $\sum f(n^{-1})$ converges we have $f(n^{-1}) \to 0$ as $n \to \infty$. Therefore $f(0) = 0$.

Since f'' is continuous on the bounded interval $[-1, 1]$, it follows that there is some contant M for which $|f''(x)| \leq M$ for $x \in [-1, 1]$. Hence

$$\sum_n \frac{f''(c_n)}{2n^2}$$

converges.

It remains to consider the second term on the RHS. If $f'(0)$ were nonzero then $f'(0) \sum n^{-1}$ would diverge, implying that $\sum_n f(n^{-1})$ also diverges. However, this contradicts the fact that $\sum_n f(n^{-1})$ converges, as given in the statement of the problem. So $f'(0) = 0$.

Therefore

$$\sum_n |f(n^{-1})| = \sum_n \frac{|f''(c_n)|}{2n^2}$$

converges.