Optimal reinsurance under VaR and CTE risk measures

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ABSTRACT

Let $X$ denote the loss initially assumed by an insurer. In a reinsurance design, the insurer cedes part of its loss, say $f(X)$, to a reinsurer, and thus the insurer retains a loss $I_f(X) = X - f(X)$. In return, the insurer is obligated to compensate the reinsurer for undertaking the risk by paying the reinsurance premium. Hence, the sum of the retained loss and the reinsurance premium can be interpreted as the total cost of managing the risk in the presence of reinsurance. Based on a technique used in [Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks. In: Willey Series in Probability and Statistics] and motivated by [Cai J., Tan K.S., 2007. Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measure. Astin Bull. 37 (1), 93–112] on using the value-at-risk (VaR) and the conditional tail expectation (CTE) of an insurer’s total cost as the criteria for determining the optimal reinsurance, this paper derives the optimal ceded loss functions in a class of increasing convex ceded loss functions. The results indicate that depending on the risk measure’s level of confidence and the safety loading for the reinsurance premium, the optimal reinsurance can be in the forms of stop-loss, quota-share, or change-loss.

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1. Introduction

Reinsurance is a commonly employed risk management strategy to ensure that the insurer’s earnings remain relatively stable or to protect the insurer against potentially large losses. Let $X$ be the (aggregate) loss initially assumed by an insurer. Suppose that $X$ is a non-negative random variable with cumulative distribution function $F_X(x) = \Pr[X \leq x]$, survival function $S_X(x) = 1 - F_X(x) = \Pr[X > x]$, and mean $0 < E[X] < \infty$. Under a reinsurance arrangement, the insurer cedes part of its loss, say $f(X)$ with $0 \leq f(X) \leq X$, to a reinsurer, and thus the insurer retains a loss $I_f(X) = X - f(X)$, where the function $f(x)$, satisfying $0 \leq f(x) \leq x$, is known as a ceded loss function and the function $I_f(x) = x - f(x)$ is called a retained loss function.

By transferring part of the risk exposure to a reinsurer, the insurer also incurs an additional cost in the form of reinsurance premium that is payable to the reinsurer. Let $\delta_f(X)$ denote the reinsurance premium which corresponds to a ceded loss function $f(x)$ and let $T_f(X)$ represent the resulting total cost or the total risk exposure of the insurer in the presence of reinsurance. Then we obtain the following relationship:

$$T_f(X) = I_f(X) + \delta_f(X).$$ \hspace{1cm} (1.1)

This demonstrates that in the presence of reinsurance, the insurer is now concerned with the risk exposure $T_f(X)$, instead of the ground up loss $X$. This also suggests that an appropriate choice of the ceded loss function can provide an effective way of reducing the risk exposure of an insurer.

In practice, there are a variety of reinsurance designs from which an insurer can choose. These include quota-share reinsurance with $f(x) = ax$ and $I_f(x) = (1 - a)x$, $0 < a \leq 1$; stop-loss reinsurance with $f(x) = (x - d)_+$, $d = \max[0, x - d]$ and $I_f(x) =$...
min\{x, d\}, d \geq 0; change-loss reinsurance with \( f(x) = a(x - d)_+ \) and \( I_f(x) = (1 - a)x + a\min\{x, d\} \). Various optimization criteria have been proposed for the determination of the optimal reinsurance. For example, it is well known that the stop-loss reinsurance is the optimal one that minimizes the variances of retained losses among the class of ceded loss functions that have the same expectations; see, for example, Bowers et al. (1997), Kaas et al. (2001) and Gerber (1979). By using the criterion of minimizing some specific convex risk measures, Gajek and Zagrodny (2004) demonstrated that change-loss design is the optimal reinsurance. In a series of published papers, Kaluszka (2001, 2004a,b, 2005) also makes important contributions to the optimal reinsurance models.

More recently, using risk measures such as the value-at-risk (VaR) and the conditional tail expectation (CTE), Cai and Tan (2007) derived explicitly the optimal retention level of a stop-loss reinsurance under the expectation premium principle. Their work, in part, was sparked by an unprecedented surge in the usage of these measures as risk management tools among banks, financial institutions, and insurance companies in recent years. See, for example, Artzner et al. (1999), Basak and Shapiro (2001), McNeil et al. (2005), Cai and Li (2005), Inui and Kijima (2005) and Yamai and Yoshida (2005).

The VaR of a non-negative random variable \( X \) at a confidence level \( 1 - \alpha, 0 < \alpha < 1 \), is defined as

\[
\text{VaR}_\alpha(X) = \inf \{x : \Pr[X > x] \leq \alpha\}.
\]

Note that if \( \alpha = 0 \) or \( \alpha = 1 \), then \( \text{VaR}_{\alpha=0}(X) = 0 \) and \( \text{VaR}_{\alpha=1}(X) = \infty \). By defining VaR and CTE as, respectively, the VaR and CTE of the total cost \( T_f(X) \) in some classes of ceded loss functions. More specifically, our objective is to seek the optimal reinsurance in the following class of ceded loss functions:

**Definition 1.1.** Let \( \mathcal{F} \) denote the class of ceded loss functions, which consists of all increasing convex functions \( f(x) \) defined on \( [0, \infty) \) and satisfying \( 0 \leq f(x) \leq x \) for \( x \geq 0 \) but excluding \( f(x) \equiv 0 \).

Mathematically, our optimal reinsurance model can be formulated as follows, depending on the chosen risk measure:

**VaR-optimization:** \( \text{VaR}_{T_f}(\alpha) = \min_{f \in \mathcal{F}} \{\text{VaR}_{T_f}(\alpha)\} \) (1.5)

and

**CTE-optimization:** \( \text{CTE}_{T_f}(\alpha) = \min_{f \in \mathcal{F}} \{\text{CTE}_{T_f}(\alpha)\} \) (1.6)

where \( \text{VaR}_{T_f}(\alpha) \) and \( \text{CTE}_{T_f}(\alpha) \) are defined analogously as in (1.2) and (1.3) except for the total risk random variable \( T_f(X) \).

To facilitate the discussion of the optimal ceded loss function in \( \mathcal{F} \) of the above optimization problems, we introduce the following class of ceded loss functions:

**Definition 1.2.** Let \( \mathcal{H} \) denote the class of ceded loss functions, which consists of all non-negative functions \( h(x) \) defined on \( [0, \infty) \) with the following form

\[
h(x) = \sum_{j=1}^{n} c_{nj}(x - d_{nj})_+ \quad x \geq 0, n = 1, 2, \ldots, \quad (1.7)
\]

where \( c_{nj} > 0 \) and \( d_{nj} \geq 0 \) are constants such that

\[
0 < \sum_{j=1}^{n} c_{nj} \leq 1 \quad \text{and} \quad 0 \leq d_{n,1} \leq d_{n,2} \leq \cdots \leq d_{n,n}
\]

for all \( n = 1, 2, \ldots \).

The importance of analyzing the optimal reinsurance in \( \mathcal{H} \) can be argued as follows. First, note that \( \mathcal{H} \) is a subclass of \( \mathcal{F} \) and all functions in \( \mathcal{F} \) are continuous on \( [0, \infty) \) since increasing convex functions are continuous. Second, by adopting a technique similar to the proof of Theorem 1.5.7 in (Müller and Stoyan (2002), p.18) on increasing convex order, we formally show that any function in \( \mathcal{F} \) is the limit of a sequence of functions in \( \mathcal{H} \), namely the class \( \mathcal{H} \) is a dense subclass of \( \mathcal{F} \) (see Lemma 3.1). Consequently, by using some convergence results on VaR and CTE, we prove that the optimal functions in \( \mathcal{H} \), which minimize the VaR and CTE of the total cost \( T_h(x) \) for \( h \in \mathcal{H} \), also optimally minimize the VaR and CTE of the total cost \( T_f(X) \) for \( f \in \mathcal{F} \). The above arguments imply that we can deduce the solutions to the optimal reinsurance models (1.5) and (1.6) by first confining the optimal ceded loss functions within the class \( \mathcal{H} \). Following this strategy, we derive the optimal reinsurance analytically as presented in Theorems 3.1 and 4.1. These results indicate that optimal reinsurance can be in the forms of stop-loss, quota-share, or change-loss, depending on the risk measure's level of confidence and the safety loading for the reinsurance premium.

Throughout this paper, the terms “increasing” and “decreasing” mean “non-decreasing” and “non-increasing”, respectively. To simplify our discussions, we assume that \( X \) has a continuous strictly increasing distribution function on \( (0, \infty) \) with a possible jump at \( 0 \), which allows \( X \) to be a random sum \( \sum_{n=1}^{\infty} X_n \) and is an important special case in actuarial loss model. See Cai and Tan (2007) for details. Furthermore, to avoid discussing trivial cases, we further assume that the parameter \( \alpha \) associated with the definitions of VaR and CTE satisfies

\[
0 < \alpha < S_X(0). \quad (1.8)
\]

The rest of the paper is organized as follows. Section 2 provides additional notations and preliminary results on the VaR of \( T_h(x) \) for \( h \in \mathcal{H} \). Sections 3 and 4 analyze the solutions to the optimal reinsurance models (1.5) and (1.6), respectively. Section 5 concludes our paper and the Appendix presents some of the technical properties, lemmas and proofs.

### 2. Preliminaries

By defining \( \text{VaR}_{T_f}(\alpha) \) and \( \text{CTE}_{T_f}(\alpha) \) as, respectively, the VaR and CTE of the retained loss random variable \( I_f(X) \), then it follows from the translation invariance property of VaR and CTE that

\[
\text{VaR}_{I_f}(\alpha) = \text{VaR}_{I_f}(\alpha) + \delta_f(X), \quad (2.1)
\]

and

\[
\text{CTE}_{I_f}(\alpha) = \text{CTE}_{I_f}(\alpha) + \delta_f(X). \quad (2.2)
\]
The above relations can also be justified as follows: (2.1) follows from (A.1) and (1.1), which in turn leads to the second relation together with the definitions of CTE.

Under our assumption that the reinsurance premium is determined using the expectation premium principle, this implies that for any function \( h(x) = \sum_{j=1}^{n} c_{nj}(x - d_{nj})^+ \in \mathcal{H} \), the reinsurance premium on the ceded loss \( h(X) \) can be written as

\[
\delta_h(X) = (1 + \rho)E[h(X)] = (1 + \rho) \left[ \sum_{j=1}^{n} c_{nj} \int_{d_{nj}}^{\infty} S_h(x) \, dx \right]. \tag{2.3}
\]

Furthermore, by defining \( A_{nj,i} = 1 - \sum_{j=1}^{n_i} c_{nj,j} \) and \( B_{nj,i} = \sum_{j=1}^{n_i} c_{nj,j} d_{nj,j}, \ i = 1, \ldots, n \), it is easy to show that the retained loss is given by

\[
I_{h}(X) = X - h(X) = X - \sum_{j=1}^{n} c_{nj}(X - d_{nj})^+.
\]

We proceed with deriving the solutions to the optimal reinsurance model

\[
\text{VaR}_{\text{opt}}(\alpha)(x) = \min_{h \in \mathcal{H}} \left\{ \text{VaR}_{\text{opt}}(h)(\alpha) \right\}.
\tag{3.1}
\]

We now present the following two lemmas. The former lemma, the proof of which we delay to the Appendix, is important for proving the latter lemma.

**Lemma 3.1.** For any \( f \in \mathcal{F} \), there exists a sequence of functions \( \{h_n, n = 1, 2, \ldots \} \) in \( \mathcal{H} \) such that \( \lim_{n \to \infty} h_n(x) = f(x) \) for all \( x \geq 0 \) and

\[
h_n(x) \leq f(x) \leq x \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad n = 1, 2, \ldots.
\]

**Lemma 3.2.** Optimal ceded loss functions which minimize the VaR of insurer’s total risk in the class \( \mathcal{H} \) are also optimal in the class \( \mathcal{F} \).

**Proof.** Let \( h^* \) be any optimal ceded loss function in the class \( \mathcal{H} \) under the VaR criterion, i.e., the solution to the optimization problem (3.1). We need to show that

\[
\text{VaR}_{\text{opt}}(h^*)(\alpha) \geq \text{VaR}_{\text{opt}}(h)(\alpha), \quad \text{for any} \quad f \in \mathcal{F}.
\tag{3.2}
\]

By Lemma 3.1, we take a sequence of functions \( \{h_n, n = 1, 2, \ldots \} \) in \( \mathcal{H} \) satisfying

\[
\lim_{n \to \infty} h_n(x) = f(x) \quad \text{for all} \quad x \geq 0
\]

and

\[
h_n(x) \leq f(x) \leq x \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad n = 1, 2, \ldots.
\tag{3.4}
\]

Then it follows from the dominated convergence theorem that

\[
\lim_{n \to \infty} \text{VaR}_{\text{opt}}(h_n)(\alpha) = \text{VaR}_{\text{opt}}(f)(\alpha),
\tag{3.5}
\]

and the optimality of \( h^* \) in the class \( \mathcal{F} \) implies

\[
\text{VaR}_{\text{opt}}(h^*)(\alpha) \geq \text{VaR}_{\text{opt}}(h)(\alpha),
\tag{3.6}
\]

for any \( n = 1, 2, \ldots \). Note that the retained loss functions \( h_n(x) = x - h_n(x) \) and \( I_n(x) = x - f(x) \) are increasing and continuous. Thus, by the continuous mapping theorem we see that \( T_{h_n}(x) = h_n(x) + \delta_{h_n}(x) = h_n(x) + (1 + \rho)E[h_n(X)] \rightarrow I_n(x) + (1 + \rho)E[f(X)] \) as \( n \to \infty \). Thus, it follows from part (b) of Proposition A.1 that \( \lim_{n \to \infty} \text{VaR}_{\text{opt}}(h_n)(\alpha) = \text{VaR}_{\text{opt}}(f)(\alpha) \), which, together with (3.6), implies (3.2) and hence we complete the proof. □

We proceed with deriving the solutions to the optimal reinsurance model (3.1). For each \( n = 1, 2, \ldots \), and a given \( 0 < \alpha < 1 \), let us first define the following sets of \( (d_{nj,n}, \ldots, d_{nj,n}) \):

\[
\begin{align*}
D_n^0 &= \{ (d_{nj,n}, \ldots, d_{nj,n}) : 0 \leq d_{nj,n} \leq \cdots \leq d_{nj,n} \}, \\
D_n^0 &= \{ (d_{nj,n}, \ldots, d_{nj,n}) : \delta X_n^1(\alpha) \leq d_{nj,n} \leq \cdots \leq d_{nj,n} \}, \\
D_n^0 &= \{ (d_{nj,n}, \ldots, d_{nj,n}) : 0 \leq d_{nj,n} \leq \cdots \leq d_{nj,n} \leq \delta X_n^1(\alpha) \}.
\end{align*}
\tag{3.7}
\]

Note that \( D_n^0, D_n^0, \ldots, D_n^0 \) form a partition of \( D_n \) and \( D_n = \bigcup_{0 \leq \alpha \leq 1} D_n^0 \). For any \( h(x) = \sum_{j=1}^{n_i} c_{nj,j} (x - d_{nj,j})^+ \in \mathcal{H} \), we use the notation \( \text{VaR}_{\text{opt}}(h)(\alpha) \) to denote the VaR of the insurer’s total risk associated with the ceded loss \( h(X) \) as a function of \( d_{nj,j} \). Combining (2.3) and (2.7), we obtain the following lemma for the explicit expressions of \( \text{VaR}_{\text{opt}}(h_n)(\alpha) \) in terms of \( d_{nj,n} \) and \( d_{nj,n} \).

See Proposition A.2 in the Appendix for some useful properties associated with functions \( g(\cdot) \) and \( u(\cdot) \).
Lemma 3.3. For any $h(x) = \sum_{j=1}^{n} c_{nj} (x - d_{nj})_+$ $\in \mathcal{H}$ and a given confidence level $1 - \alpha$ with $0 < \alpha < S_{X}(0)$:

(a) When $S_{X}^{-1}(\alpha) \leq d_{n,1}$, i.e., $(d_{n,1}, \ldots, d_{n,n}) \in D_{n}^{l}$,

$$\text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha) = S_{X}^{-1}(\alpha) + \frac{1}{\rho} \left( \sum_{j=1}^{n} c_{nj} \int_{d_{nj}}^{\infty} S_{X}(x)dx \right).$$

(b) When $d_{nj} \leq S_{X}^{-1}(\alpha) \leq d_{nj+1}$, i.e., $(d_{n,1}, \ldots, d_{n,n}) \in D_{n}^{l}$, and for $i = 1, \ldots, n - 1$,

$$\text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha) = \left( 1 - \sum_{j=1}^{n} c_{nj} \right) S_{X}^{-1}(\alpha) + \frac{1}{\rho} \sum_{j=1}^{n} c_{nj} \int_{d_{nj}}^{\infty} S_{X}(x)dx.$$

(c) When $d_{n,n} \leq S_{X}^{-1}(\alpha)$, i.e., $(d_{n,1}, \ldots, d_{n,n}) \in D_{n}^{r}$,

$$\text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha) = \left( 1 - \sum_{j=1}^{n} c_{nj} \right) S_{X}^{-1}(\alpha) + \sum_{j=1}^{n} c_{nj} g(d_{nj}). \square$$

Based on the above expressions of $\text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$, we analyze its minimum on set $D_{n}$ for $n = 1, 2, \ldots$ by discussing its infimum on $D_{n}^{l}$ for each $i = 0, 1, 2, \ldots, n$. The results are summarized in the following lemma (see the Appendix for the proof):

Lemma 3.4. Given a confidence level $1 - \alpha$ with $0 < \alpha < S_{X}(0)$, for any function $h(x) = \sum_{j=1}^{n} c_{nj} (x - d_{nj})_+$ $\in \mathcal{H}$ with given coefficients $c_{nj}, j = 1, \ldots, n$:

(a) If

$$\rho^{*} < S_{X}(0) \quad \text{and} \quad S_{X}^{-1}(\alpha) \geq u(\rho^{*}),$$

then

$$\min_{D_{n}} \text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$$

$$= S_{X}^{-1}(\alpha) + \sum_{j=1}^{n} c_{nj} \left[ u(\rho^{*}) - S_{X}^{-1}(\alpha) \right]$$

and the minimum VaR is attained at $d_{n,1} = \ldots = d_{n,n} = d^{*}$ or at

$$h^{*}(x) = \sum_{j=1}^{n} c_{nj} (x - d^{*})_+.$$

(b) If

$$\rho^{*} \geq S_{X}(0) \quad \text{and} \quad S_{X}^{-1}(\alpha) \geq g(0),$$

then

$$\min_{D_{n}} \text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$$

$$= S_{X}^{-1}(\alpha) + \sum_{j=1}^{n} c_{nj} \left[ g(0) - S_{X}^{-1}(\rho^{*}) \right]$$

and the minimum VaR is attained at $d_{n,1} = \ldots = d_{n,n} = 0$ or at

$$h^{*}(x) = \sum_{j=1}^{n} c_{nj} x.$$

(c) For all other cases, $\min_{D_{n}} \text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$ does not exist.

We are now ready to present the key results of this section which are stated in Theorem 3.1. The above lemma is used to obtain the solution to (3.1) by comparing the minimum of $\text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$ on $D_{n}$ for each $n = 1, 2, \ldots$. Lemma 3.2, in turn, asserts that these solutions are also the solutions to the proposed optimal reinsurance model (1.5).

Theorem 3.1. For a given confidence level $1 - \alpha$ with $0 < \alpha < S_{X}(0)$:

(a) If $\rho^{*} < S_{X}(0)$ and $S_{X}^{-1}(\alpha) > u(\rho^{*})$, then $\min_{\alpha \in \mathcal{F}} \text{Var}_{\mathcal{T}_{j}}(\alpha) = u(\rho^{*})$ and the minimum VaR is attained at

$$f^{*}(x) = (x - d^{*})_+.$$

(b) If $\rho^{*} < S_{X}(0)$ and $S_{X}^{-1}(\alpha) = u(\rho^{*})$, then $\min_{\alpha \in \mathcal{F}} \text{Var}_{\mathcal{T}_{j}}(\alpha)$

$$= S_{X}^{-1}(\alpha) \quad \text{and} \quad \text{the minimum VaR is attained at}$$

$$f^{*}(x) = c (x - d^{*})_+.$$

(c) For any constant $c$ such that $0 < c \leq 1$.

(d) If $\rho^{*} \geq S_{X}(0)$ and $S_{X}^{-1}(\alpha) > g(0)$, then $\min_{\alpha \in \mathcal{F}} \text{Var}_{\mathcal{T}_{j}}(\alpha) = g(0)$ and the minimum VaR is attained at

$$f^{*}(x) = x.$$

(e) If $\rho^{*} \geq S_{X}(0)$ and $S_{X}^{-1}(\alpha) = g(0)$, then $\min_{\alpha \in \mathcal{F}} \text{Var}_{\mathcal{T}_{j}}(\alpha) = S_{X}^{-1}(\alpha) \quad \text{and} \quad \text{the minimum VaR is attained at}$$

$$f^{*}(x) = cx$$

for any constant $c$ such that $0 < c \leq 1$.

Proof. Note that all ceded loss functions given in the theorem are included in $\mathcal{H}$. Hence, by Lemma 3.2, we only need to show the optimality of these ceded loss functions in class $\mathcal{H}$; i.e., for any $h(x) = \sum_{j=1}^{n} c_{nj} (x - d_{nj})_+ \in \mathcal{H}$,

$$\text{Var}_{\mathcal{T}_{j}}(\alpha) \geq \text{Var}_{\mathcal{T}_{j}}(\alpha).$$

(a) By (3.12) of Lemma 3.4 and the assumption $S_{X}^{-1}(\alpha) > u(\rho^{*})$, we have

$$\text{Var}_{\mathcal{T}_{j}}(\alpha) \geq \min_{D_{n}} \text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$$

$$= S_{X}^{-1}(\alpha) + \sum_{j=1}^{n} c_{nj} \left[ u(\rho^{*}) - S_{X}^{-1}(\alpha) \right]$$

$$\geq S_{X}^{-1}(\alpha) + u(\rho^{*}) - S_{X}^{-1}(\alpha) = u(\rho^{*}).$$

Furthermore, by (3.12) and (3.13), $\text{Var}_{\mathcal{T}_{j}}(\alpha) = u(\rho^{*})$ when $\sum_{j=1}^{n} c_{nj} = 1$ in (3.13) or when $f^{*}(x) = (x - d^{*})_+$. Hence, (3.22) implies (3.21).

(b) By (3.12) of Lemma 3.4 and the assumption $S_{X}^{-1}(\alpha) = g(0)$, we have

$$\text{Var}_{\mathcal{T}_{j}}(\alpha) \geq \min_{D_{n}} \text{Var}_{\mathcal{T}_{j}}(d_{n,1}, \ldots, d_{n,n}, \alpha)$$

$$= S_{X}^{-1}(\alpha) + \sum_{j=1}^{n} c_{nj} \left[ g(0) - S_{X}^{-1}(\rho^{*}) \right]$$

$$\geq S_{X}^{-1}(\alpha) + g(0) = S_{X}^{-1}(\alpha).$$

Furthermore, by (3.12) and (3.13), $\text{Var}_{\mathcal{T}_{j}}(\alpha) = S_{X}^{-1}(\alpha)$ when $\sum_{j=1}^{n} c_{nj} = c$ in (3.13) or when $f^{*}(x) = c (x - d^{*})_+$ for any $0 < c \leq 1$. Hence, (3.23) implies (3.21).

Parts (c) and (d) are proved similarly as in Parts (a) and (b) by using (3.15) of Lemma 3.4 and the assumptions $S_{X}^{-1}(\alpha) > g(0)$ and $S_{X}^{-1}(\alpha) = g(0)$, respectively. \(\square\)
Remark 3.1. Theorem 3.1 establishes that for our proposed optimal reinsurance model, the optimal reinsurance is a stop-loss reinsurance in case (a), a change-loss reinsurance in case (b), and a quota-share reinsurance in cases (c) and (d).

We conclude this section by examining several special functions in $\mathcal{F}$, as illustrated in the following examples:

Example 3.1. Let $f(X) = X$ so that an insurer cedes all its losses to a reinsurer. In this case, the retained loss is $I_f(X) = 0$. The total cost for the insurer is $T_f(X) = (1 + \rho)E[X]$. However, the VaR of a constant is itself. Hence, $\text{VaR}_{T_f}(x) = (1 + \rho)E[X] = g(0)$. It is easy to verify that

$$g(0) = (1 + \rho)E[X] > u(\rho^*)$$

Indeed, noticing that $S_X(x)$ is decreasing in $x$ and $S_X(d^*) = S_X(\int_0^d S_X(x)dx + \int_d^\infty S_X(x)dx)$

$$> S_X^{-1}(\rho^*) + \frac{1}{\rho^*} \int_0^{\infty} S_X(x)dx = u(\rho^*)$$

Hence, $\text{VaR}_{T_f}(x)$ is bigger than the minimum VaR in cases (a) and (b) of Theorem 3.1. However, $f$ is optimal in cases (c) and (d). $\square$

Example 3.2. Let $f_d(X) = aX$ for $0 < a < 1$ so that an insurer uses a quota-share reinsurance. In this case, the retained loss is $I_f(X) = (1 - a)X$. The total cost for the insurer is $T_f(X) = (1 - a)X + a(1 + \rho)E[X]$. By (4.1), we have

$$\text{VaR}_{T_f}(x) = (1 - a)S_X^{-1}(\rho^*) + a(1 + \rho)E[X]$$

It follows from (3.24) that in cases (a) and (b) of Theorem 3.1, $\text{VaR}_{T_f}(x) = (1 - a)u(\rho^*) + au(\rho^*) = u(\rho^*)$. In case (c), $\text{VaR}_{T_f}(x) = (1 - a)g(0) + ag(0) = g(0)$. However, $f_d$ is optimal in case (d). $\square$

Example 3.3. Let $f_d(X) = (X - d)^+$ for $d > 0$, namely, an insurer uses a stop-loss reinsurance. In this case, the retained loss is $I_f(X) = \min[X, d]$. The total cost for the insurer is $T_f(X) = \min[X, d] + (1 + \rho)E[X - d]^+$. It has been proved in Cai and Tan (2007) under the conditions of case (a) of Theorem 3.1 that $\text{VaR}_{T_f}(x) = u(\rho^*)$ and the minimum VaR is attained at $d^* = S_X^{-1}(\rho^*)$. $\square$

4. Optimal reinsurance under CTE risk measure

The last section derived explicitly the optimal reinsurance that minimizes the VaR of the total cost of an insurer for reinsuring its losses. In this section, we extend the analysis by assuming CTE as the relevant risk measure. More specifically, we are interested in the optimal ceded loss functions for the CTE-optimization as formulated in (1.6). Our procedure to identify the optimal ceded functions under the CTE criterion will be parallel to the situation of VaR criterion in the previous section. We will also establish the fact that optimal ceded loss functions in the class $\mathcal{H}$, which minimize the CTE of the insurer’s total cost, are also optimal in the larger class $\mathcal{F}$ (see Lemma 4.2), and then consider the following optimization problem over the class of ceded loss functions in $\mathcal{F}$:

$$\text{CTE}_{I_f, T}(x) = \min_{h \in \mathcal{H}} \left[ \text{CTE}_{T_f}(x) \right]$$

It should be emphasized that it is considerably more complicated to discuss the optimal ceded loss functions under the CTE criterion than the VaR criterion. The difficulty lies mainly in the proof of the following lemma (Lemma 4.1), which will be crucial to the proof of Lemma 4.2. We defer the proof of Lemma 4.1 to the Appendix.

Lemma 4.1. For any $f(x) \in \mathcal{F}$, there exists a sequence of functions $h_n(x) \in \mathcal{H}$ such that

$$\lim_{n \to \infty} h_n(x) \to f(x) \quad \text{for all } x \geq 0$$

and for $\alpha < S_t(0)$,

$$\Pr[I_{h_n}(X) \geq \text{VaR}_{h_n}(x)(\alpha)] \to \Pr[I_f(X) \geq \text{VaR}_{I_f}(x)(\alpha)]$$

as $n \to \infty$. (4.3)

Lemma 4.2. Optimal ceded loss functions which minimize the CTE of the insurer’s total risk in class $\mathcal{H}$ are also optimal in class $\mathcal{F}$. Proof. Let $h^*$ be the optimal ceded loss function in class $\mathcal{H}$ and $f$ be any ceded loss function in the class $\mathcal{F}$. We need to show

$$\text{CTE}_{I_f(\alpha)}(x) \geq \text{CTE}_{h^*}(x)(\alpha).$$

Recall that by combining (A.3) of Proposition A.1 and (2.1), we have

$$\text{CTE}_{h^*}(x)(\alpha) = \text{VaR}_{h^*}(x)(\alpha)$$

and

$$\text{CTE}_{I_f(\alpha)}(x) = \text{VaR}_{I_f}(x)(\alpha)$$

By Lemma 4.1, there exists a sequence of functions $h_n(x) \in \mathcal{H}$ such that

$$\lim_{n \to \infty} h_n(x) = f(x) \quad \text{with } 0 \leq h_n(x) \leq f(x) \quad \text{for any } x \geq 0$$

and

$$\Pr[I_{h_n}(X) \geq \text{VaR}_{h_n}(x)(\alpha)] \to \Pr[I_f(X) \geq \text{VaR}_{I_f}(x)(\alpha)], \quad \text{as } n \to \infty.$$ (4.7)

Thus, using the dominated convergence theorem yields

$$\lim_{n \to \infty} E[I_{h_n}(X)] = E[I_f(X)],$$

since $0 < EX < \infty$. Recall that $I_f(x) = x - f(x)$ is increasing and continuous in $x \geq 0$ and hence it follows from part (b) of Proposition A.1 that

$$\lim_{n \to \infty} \text{VaR}_{h_n}(x)(\alpha) = \text{VaR}_{I_f}(x)(\alpha).$$ (4.9)

Now, by denoting $Y_{h_n} = I_n(X) - \text{VaR}_{h_n}(x)(\alpha)$ and the continuity of function $\kappa(y) = y \mathbb{1}_{[y \geq 0]}$, we have $\kappa(Y_{h_n} \xrightarrow{a.s.} \kappa(Y_f)$). Hence, based on the fact that

$$0 \leq \nu(Y_{h_n}) = [X - \text{VaR}_{h_n}(x)(\alpha)]I_{[X - \text{VaR}_{h_n}(x)(\alpha) \geq 0]} \leq X, \quad \text{a.s.,}$$

we conclude

$$\lim_{n \to \infty} \int_{[X - \text{VaR}_{h_n}(x)(\alpha) \geq 0]} S_{I_n}(x)dx = \lim_{n \to \infty} E[\kappa(Y_{h_n})] = E[\kappa(Y_f)] = \int_{[X - \text{VaR}_{I_f}(x)(\alpha) \geq 0]} S_{I_f}(x)dx.$$ (4.10)

using the dominated convergence theorem. Consequently, by (4.5)-(4.10), we obtain

$$\lim_{n \to \infty} \text{CTE}_{h_n}(x)(\alpha) = \text{CTE}_{I_f}(x)(\alpha).$$ (4.11)
On the other hand, the optimality of \( h^\ast \) in \( \mathcal{H} \) implies
\[
\text{CTE}_{F_n}(x) \geq \text{CTE}_{F_n}(x),
\]
(4.12)
for \( n = 1, 2, \ldots \), which together with (4.11) implies (4.4). \( \square \)

To proceed with the discussion of the solution to the optimal reinsurance model (4.1), it is convenient to first determine the explicit expressions for the corresponding CTE of the total cost. Recall that from (A.3) of Proposition A.1 and together with (2.1), we obtain
\[
\text{CTE}_f(x) = \text{Var}_f(x) + \frac{1}{\alpha} \int_{\text{Pr}[f_I(x) \geq \text{Var}_f(x)]}^\infty S_f(x) \, dx.
\]
(4.13)

Similarly to the VaR-optimization, for any function \( h(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+ \in \mathcal{H} \), we express the CTE of the ceded loss \( h(x) \) as a function of \( d_{n1}, \ldots, d_{nn} \), which we denote as \( \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) \). Then, by defining the following function
\[
v(x) = S_{X}^{-1}(x) + \frac{1}{\alpha} \int_{S_{X}(t)}^\infty S_X(t) \, dt, \quad x \geq 0,
\]
(4.14)
we obtain explicit expressions of \( \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) \), as shown in Lemma 4.3. Capitalizing on this lemma, we determine the minimum of \( \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) \) on the set \( D_n \) for each \( i = 0, 1, 2, \ldots, n \) and a given integer \( n \). The results are given in Lemma 4.4. The proofs of these two lemmas are presented in the Appendix.

**Lemma 4.3.** Given a confidence level \( 1 - \alpha \) with \( 0 < \alpha < S_X(0) \), for any \( h(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+ \in \mathcal{H} \):

(a) When \( S_{X}^{-1}(x) \leq d_{ni} \), i.e., \( (d_{n1}, \ldots, d_{nn}) \in D_n^0 \),
\[
\text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) = v(\alpha) + \frac{1}{\rho^\ast} - \frac{1}{\alpha} \sum_{j=1}^n c_{nj} \int_{d_{nj}}^\infty S_X(x) \, dx.
\]
(4.15)

(b) When \( d_{ni} \leq S_{X}^{-1}(x) \leq d_{ni+1} \), i.e., \( (d_{n1}, \ldots, d_{nn}) \in D_n^i \) for \( i = 1, \ldots, n - 1 \),
\[
\text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) = \left( 1 - \sum_{j=1}^i c_{nj} \right) v(\alpha)
\]
\[
+ \sum_{j=1}^i c_{nj} g(d_{nj}) + \frac{1}{\rho^\ast} - \frac{1}{\alpha} \sum_{j=i+1}^n c_{nj} \int_{d_{nj}}^\infty S_X(x) \, dx.
\]
(4.16)

(c) When \( d_{nn} \leq S_{X}^{-1}(x) \), i.e., \( (d_{n1}, \ldots, d_{nn}) \in D_n^n \),
\[
\text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) = \left( 1 - \sum_{j=1}^n c_{nj} \right) v(\alpha)
\]
\[
+ \sum_{j=1}^n c_{nj} g(d_{nj}).
\]
(4.17)

**Lemma 4.4.** Consider any function \( h(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+ \in \mathcal{H} \) with given coefficients \( c_{nj}, j = 1, \ldots, n \), and a confidence level \( 1 - \alpha \) such that \( 0 < \alpha < S_X(0) \).

(a) If \( \alpha < \rho^\ast < S_X(0) \), then
\[
\min_{D_n} \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) = v(\alpha) + \sum_{j=1}^n c_{nj} [u(\rho^\ast) - v(\alpha)]
\]
and the minimum CTE is attained at \( d_{n1} = \cdots = d_{nn} = d^\ast \) or at
\[
h^\ast(x) = \sum_{j=1}^n c_{nj}(x - d^\ast)_+.
\]
(4.19)

(b) If \( \alpha = \rho^\ast < S_X(0) \), then
\[
\min_{D_n} \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) = v(\alpha)
\]
(4.20)
and the minimum CTE is attained at any \( (d_{n1}, \ldots, d_{nn}) \) or at
\[
h^\ast(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+
\]
(4.21)
provided that \( (d_{n1}, \ldots, d_{nn}) \) satisfy \( d^\ast \leq d_{n1} \leq \cdots \leq d_{nn} \). In particular, the minimum CTE is attained at \( (d^\ast_1, \ldots, d^\ast_n) \) or at \( h^\ast(x) \).

(c) If \( \alpha < S_X(0) \leq \rho^\ast \), then
\[
\min_{D_n} \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha)
\]
\[
= v(\alpha) + \sum_{j=1}^n c_{nj}[g(0) - v(\alpha)]
\]
(4.22)
and the minimum CTE is attained at \( d_{n1} = \cdots = d_{nn} = 0 \) or at
\[
h^\ast(x) = \sum_{j=1}^n c_{nj}x^j.
\]
(4.23)

(d) For all other cases, \( \min_{D_n} \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) \) does not exist.

Finally, by comparing the minimum of \( \text{CTE}_{F_n}(d_{n1}, \ldots, d_{nn}, \alpha) \) for each \( n \), we obtain the solutions to the optimization problem (4.1). Lemma 4.2, in turn, asserts that these solutions are also the optimal ceded loss functions in the class \( \mathcal{F} \) and hence the solutions to our proposed optimal reinsurance model (1.6). The results are summarized in the following theorem.

**Theorem 4.1.** For a given confidence level \( 1 - \alpha \) with \( 0 < \alpha < S_X(0) \):

(a) If \( \alpha < \rho^\ast < S_X(0) \), then \( \min_{h \in \mathcal{H}} \text{CTE}_{f}(h) = u(\rho^\ast) \) and the minimum CTE is attained at
\[
f^\ast(x) = (x - d^\ast)_+.
\]
(4.24)

(b) If \( \alpha = \rho^\ast < S_X(0) \), then \( \min_{h \in \mathcal{H}} \text{CTE}_{f}(h) = u(\rho^\ast) \) and the minimum CTE is attained at
\[
f^\ast(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+ \in \mathcal{H}
\]
(4.25)
such that \( d^\ast \leq d_{n1} \leq \cdots \leq d_{nn} \) and \( n = 1, 2, \ldots \).

(c) If \( \alpha < S_X(0) \leq \rho^\ast \), then \( \min_{h \in \mathcal{H}} \text{CTE}_{f}(h) = u(\rho^\ast) \) and the minimum CTE is attained at
\[
f^\ast(x) = x.
\]
(4.26)

**Proof.** The proof is based on Lemmas 4.2 and 4.4 and parallels the proof of Theorem 3.1. Note that all ceded loss functions given in the theorem are included in the class \( \mathcal{H} \). Hence, by Lemma 4.2, we only need to show the optimality of these ceded loss functions in the class \( \mathcal{F} \); i.e., for any \( h(x) = \sum_{j=1}^n c_{nj}(x - d_{nj})_+ \in \mathcal{H} \),
\[
\text{Var}_f(x) \geq \text{Var}_{f_{\rho^\ast}}(x).
\]
(4.27)

(a) First, note that when \( \alpha < \rho^\ast < S_X(0) \), we have
\[
u(\rho^\ast) < v(\alpha).
\]
(4.28)

Indeed, in this case, \( S_{X}^{-1} > S_{X}^{-1}(\rho^\ast) = d^\ast \), which, together with (A.4), implies \( g(S_{X}^{-1}(\alpha)) \geq g(d^\ast) \). Moreover, it follows from the definitions of the functions \( u(\cdot), v(\cdot), \) and \( g(\cdot) \) that \( v(\alpha) > u(\alpha) = g(S_{X}^{-1}(\alpha)) \) and \( g(d^\ast) = u(\rho^\ast) \). Consequently, we obtain (4.28),
which, together with (4.18) of Lemma 4.4, implies for any function \( h \in \mathcal{H} \),
\[
\text{CTE}_{T_{d}(X)}(\alpha) \geq \min_{D_{m}} \text{CTE}_{T_{d}(X)}(d_{n, 1}, \ldots, d_{n, n}, \alpha)
\]
\[
= v(\alpha) + \sum_{j=1}^{n} c_{n,j}[u(\rho^{*}) - v(\alpha)] \geq u(\rho^{*}).
\] (4.29)

Furthermore, (4.18) and (4.19) lead to \( \text{CTE}_{T_{d}(X)}(\alpha) \geq u(\rho^{*}) \) when \( \sum_{j=1}^{n} c_{n,j} = 1 \) in (4.19) or when \( f^{*}(x) = (x - d^{*})^{+} \). Hence, (4.29) implies (4.27).

(b) By (4.20) of Lemma 4.4, we have for any function \( h \in \mathcal{H} \),
\[
\text{CTE}_{T_{d}(X)}(\alpha) \geq \min_{D_{m}} \text{CTE}_{T_{d}(X)}(d_{n, 1}, \ldots, d_{n, n}, \alpha) = v(\alpha).
\] (4.30)

Furthermore, by (4.20) and (4.21), \( \text{CTE}_{T_{d}(X)}(\alpha) = v(\alpha) \) when \( f^{*} \) is of the form such that \( d^{*} \leq d_{n, 1} \leq \cdots \leq d_{n, n} \), and \( n = 1, 2, \ldots \). Hence, (4.30) implies (4.27).

(c) By the definitions of \( v(\cdot) \) and \( g(\cdot) \), we have \( g(S_{n}^{-1}(\alpha)) < v(\alpha) \) and by (A.6) of Proposition A.2, we have \( g(0) \leq g(S_{n}^{-1}(\alpha)) \). Thus, \( g(0) < v(\cdot) \), which together with (4.22) of Lemma 4.4, leads to
\[
\text{CTE}_{T_{d}(X)}(\alpha) \geq \min_{D_{m}} \text{CTE}_{T_{d}(X)}(d_{n, 1}, \ldots, d_{n, n}, \alpha)
\]
\[
= v(\alpha) + \sum_{j=1}^{n} c_{n,j}[g(0) - v(\alpha)] \geq v(\alpha),
\] (4.31)

for any function \( h \in \mathcal{H} \). Furthermore, by (4.22) and (4.23), \( \text{CTE}_{T_{d}(X)}(\alpha) = v(\alpha) \) when \( \sum_{j=1}^{n} c_{n,j} = 1 \) in (4.23) or when \( f^{*}(x) = x \). Hence, (4.31) implies (4.27). \( \square \)

5. Conclusion

It is well known that reinsurance can be an effective risk management technique for insurers to transfer part of its risk to the reinsurer. The key, however, hinges on an optimal choice of reinsurance contracts. By formulating an optimization problem that minimizes the VaR (or CTE) of the total cost of the reinsurer, this paper establishes the conditions for optimal reinsurance designs. In particular, depending on the confidence level \( 1 - \alpha \) for the risk measure and the safety loading \( \rho \) for the reinsurance premium, we formally justified that in some cases, a stop-loss reinsurance is optimal while in some other cases, a quota-share reinsurance or a change-loss reinsurance is optimal. It is also of interest to note that the conditions for optimal solutions for CTE are less restrictive than those for VaR.

It should be pointed out that a basic assumption in our proposed optimal reinsurance model is adopting the expectation premium principle for setting the reinsurance premium. In practice, there exists a number of other premium principles. It will be of interest to investigate the impact of these premium principles on the optimal reinsurance designs. We leave this for future research exploration.

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Appendix

This appendix collects the proofs of some of the results stated in the previous sections. Other necessary propositions and lemmas are also presented and proved in this appendix.

Proposition A.1. (a) If function \( f \) is increasing and continuous, then
\[
\text{VaR}_{X}(\alpha) = f(\text{VaR}_{R}(\alpha)).
\] (A.1)

(b) If functions \( f_{0} \) and \( f \) are increasing and continuous and satisfy
\[
\lim_{n \to \infty} f_{0}(x) = f(x) \text{ for all } x \geq 0,
\]
then
\[
\lim_{n \to \infty} \text{VaR}_{X}(\alpha) = \text{VaR}_{R}(\alpha).
\] (A.2)

(c) CTE and VaR of \( X \) are related as
\[
\text{CTE}_{X}(\alpha) = E[X|X \geq \text{VaR}_{R}(\alpha)]
\]
\[
= \text{VaR}_{R}(\alpha) + \frac{\int_{0}^{\infty} S_{X}(x) dx}{\Pr(X \geq \text{VaR}_{R}(\alpha))}.
\] (A.3)

Proof. Parts (a) and (b) follow immediately from the definition of VaR while for part (c), see Cai and Tan (2007). \( \square \)

Proposition A.2. (a) If \( \rho^{*} < S_{X}(0) \), then the continuous function \( g(x) \) defined in (2.9) is decreasing on \( (0, d^{*}) \) while increasing on \( (d^{*}, \infty) \) and satisfies
\[
\min_{0 \leq x \leq a} g(x) = g(a) \quad \text{for } 0 \leq a \leq d^{*},
\] (A.4)

\[
\min_{0 \leq x \leq a} g(x) = g(d^{*}) = u(\rho^{*}) \quad \text{for } d^{*} \leq a.
\] (A.5)

(b) If \( \rho^{*} \geq S_{X}(0) \), then the continuous function \( g(x) \) defined in (2.9) is increasing on \( (0, \infty) \) and satisfies
\[
\min_{0 \leq x \leq a} g(x) = g(0) \quad \text{for } a \geq 0.
\] (A.6)

Proof. (a) The proof follows from the condition that if \( \rho^{*} < S_{X}(0) \), then \( g'(x) < 0 \) for \( 0 < x < d^{*} \) and \( g'(x) > 0 \) for \( x > d^{*} \).

(b) The proof follows from \( g'(x) > 0 \) for \( x > 0 \) if \( \rho^{*} \geq S_{X}(0) \). \( \square \)

Proof of Lemma 3.1. It is known that for any non-negative decreasing convex function \( f \) defined on \( [0, \infty) \), there exists a sequence of non-negative functions \( \{h_{n}, n = 1, 2, \ldots \} \) defined on \( [0, \infty) \) such that \( h_{n}(x) = \sum_{j=1}^{n} c_{n,j} (x - d_{n,j})^{+} \) for some constants \( c_{n,j} \geq 0 \) and \( d_{n,j} \geq 0 \) and \( \lim_{n \to \infty} h_{n}(x) = f(x) \) from below for any \( x \geq 0 \), which implies that
\[
h_{n}(x) \leq f(x) \text{ for all } x \geq 0 \quad \text{and } n = 1, 2, \ldots.
\] (A.7)

See, for example, Case 1 of the proof of Theorem 1.5.7 of Müller and Stoyan (2002), p 18.

For any \( f \in \mathcal{F} \), we have \( 0 \leq f(x) \leq x \), which, together with (A.7), implies (3.4). Furthermore, it follows from (3.4) that for any \( x > 0 \) and \( n = 1, 2, \ldots \),
\[
0 \leq \frac{h_{n}(x)}{x} = \sum_{j=1}^{n} c_{n,j} (x - d_{n,j})^{+} \leq 1.
\] (A.8)

Thus, letting \( x \to \infty \) in (A.8) yields \( 0 \leq \sum_{j=1}^{n} c_{n,j} \leq 1 \) for all \( n = 1, 2, \ldots \).

Finally, to show \( h_{n} \in \mathcal{H} \), we just need to verify that \( c_{n,1} > 0, \ldots, c_{n,n} > 0 \) and \( 0 \leq d_{n,1} \leq \cdots \leq d_{n,n} \) for all \( n = 1, 2, \ldots \).

Since \( f \in \mathcal{F} \) and \( f(0) = 0 \) does not hold, there exists a positive integer \( n_{0} \) so that when \( n \geq n_{0} \), there is at least one of \( c_{n,1}, \ldots, c_{n,n} \) which is positive. Hence, for any \( n \geq n_{0} \), we delete the term \( c_{n,j} (x - d_{n,j})^{+} \) from \( h_{n}(x) \) when \( c_{n,j} = 0 \), and relabel the rest of the coefficients of \( (c_{n,j}, d_{n,j}) \) such that \( d_{n,j} \) is in an increasing order. Thus, the sequence of the functions \( \{h_{n}, n \geq n_{0}\} \) satisfies the requirements for the lemma and this completes the proof of the lemma. \( \square \)
Proof of Lemma 3.4. We will first establish the fact that $\text{VaR}_{\alpha}(X)$ $(d_1, \ldots, d_n, \alpha)$ has minimum on $D_n$ if and only if

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} \leq \min_{i=0,1,\ldots,n-1} \left\{ \inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} \right\}.$$  \hspace{3.65cm} \text{(A.9)}

To this end, we consider the value of $\inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \}$ if it is attainable on the indicated set $D_n^i$ for $i = 0, 1, 2, \ldots, n$. Note that $f_d^\infty S(x) dx$ is a decreasing function in $d \geq 0$ and $f_d^\infty S(x) dx \to 0$ as $d \to \infty$. Thus, it follows from (3.8) that

$$\inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(\infty, \ldots, \infty, \alpha) = S_X^{-1}(\alpha).$$  \hspace{2.55cm} \text{(A.10)}

For $D_n^i$, $i = 1, 2, \ldots, n$, we consider the following three possible situations:

(i) When $\rho^* < S_X(0)$ and $d^* = S_X^{-1}(\rho^*) < S_X^{-1}(\alpha)$, then from (A.5) of Proposition A.2, (3.9) and (3.10), we have for $i = 1, \ldots, n-1$,

$$\inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(d^*, \ldots, d^*, \infty, \ldots, \infty, \alpha) = \left( 1 - \sum_{j=1}^i c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^i c_{n,j} g(d^*)$$

and for $i = n$,

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(d^*, \ldots, d^*, \alpha) = \left( 1 - \sum_{j=1}^n c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^n c_{n,j} g(d^*).$$  \hspace{2.55cm} \text{(A.11)}

(ii) If $\rho^* < S_X(0)$ and $d^* = S_X^{-1}(\rho^*) < S_X^{-1}(\alpha)$, then from (A.4) of Proposition A.2, (3.9) and (3.10), we have for $i = 1, \ldots, n-1$,

$$\inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(S_X^{-1}(\alpha), \ldots, S_X^{-1}(\alpha), \infty, \ldots, \infty, \alpha) \Rightarrow \left( 1 - \sum_{j=1}^i c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^i c_{n,j} g(S_X^{-1}(\alpha))$$

and for $i = n$,

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(S_X^{-1}(\alpha), \ldots, S_X^{-1}(\alpha), \alpha) = \left( 1 - \sum_{j=1}^n c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^n c_{n,j} g(S_X^{-1}(\alpha)).$$  \hspace{2.55cm} \text{(A.12)}

(iii) When $\rho^* \geq S_X(0)$, then by Proposition A.2 (b), (3.9) and (3.10), we have for $i = 1, \ldots, n-1$,

$$\inf_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(0, \ldots, 0, \infty, \ldots, \infty, \alpha) = \left( 1 - \sum_{j=1}^i c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^i c_{n,j} g(0)$$

and for $i = n$,

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \text{VaR}_{\alpha}(X)(0, \ldots, 0, \alpha) = \left( 1 - \sum_{j=1}^n c_{n,j} \right) S_X^{-1}(\alpha) + \sum_{j=1}^n c_{n,j} g(0).$$  \hspace{2.55cm} \text{(A.13)}

Combining all the above, we immediately see that for any $h(x) = \sum_{j=1}^n c_{n,j} (x - d_{n,j})$, $e \in \mathcal{H}$ with fixed coefficients $c_{n,j}, j = 1, \ldots, n$, $\text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha)$ has a minimum only on $D_n^0$ and has infimum but no minimum on all other sets $D_n^1, D_n^2, \ldots, D_n^{n-1}$. This implies that $\text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha)$ has a minimum on $D_n$ if and only if (A.9) holds.

Note that when the above inequality (A.9) holds then

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \}$$

and $D_n = \bigcup_{i=1}^n D_n^i$. In addition, we point out that

$$g(0) = \frac{1}{\rho^*} \mathbb{E}[X] > \frac{1}{\rho^*} \int_0^{S_X^{-1}(\alpha)} S_X(x) dx > \frac{\alpha}{\rho^*} S_X^{-1}(\alpha)$$

and $u(\rho^*) > S_X^{-1}(\rho^*)$. Hence, if $\alpha \geq \rho^*$, then $g(0) > S_X^{-1}(\alpha)$ and $u(\rho^*) > S_X^{-1}(\alpha)$. These discussions are useful in proving our results in the lemma. Now we are ready to prove the results.

(a) Together with (2.10), the condition of $S_X^{-1}(\alpha) \geq u(\rho^*)$ implies the inequality $d^* = S_X^{-1}(\rho^*) < S_X^{-1}(\alpha)$. Thus, if (3.11) holds, noticing that $0 < \sum_{j=1}^{i-1} c_{n,j} < \sum_{j=1}^n c_{n,j} \leq 1$ for $i = 1, \ldots, n-1$, we see that the minimum in (A.12) is less than or equal to any of the infimums in (A.10) and (A.11) since $u(\rho^*) > S_X^{-1}(\alpha) \leq 0$. This implies that inequality (A.9) holds and so does (A.17). Consequently, it follows from (A.12) that

$$\min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \} = \min_{D_n} \{ \text{VaR}_{\alpha}(X)(d_1, \ldots, d_n, \alpha) \}$$

$$= \text{VaR}_{\alpha}(X)(d^*, \ldots, d^*, \alpha) = S_X^{-1}(\alpha) + \sum_{j=1}^n c_{n,j} [u(\rho^*) - S_X^{-1}(\alpha)].$$

where $\text{VaR}_{\alpha}(X)(d^*, \ldots, d^*, \alpha)$ means that the minimum VaR is attained at $d_{n,1} = \cdots = d_{n,n} = d^*$, and this proves (3.13).
and does not hold \( \Phi \). Proposition A.2 and with

\[
\int_{\infty}^{\infty} S_{\Phi}(x) dx = \int_{A_n, s} \int_{\infty}^{\infty} S_{\Phi} \left( \frac{x - B_{n,i}}{A_{n,i}} \right) dx = A_{n,n} \int_{\infty}^{\infty} S_{\Phi} (x) dx.
\]

Note that \( 0 \leq A_{n,n} < 1 \). If \( A_{n,n} = 0 \), then \( S_{\Phi} \left( \frac{x - B_{n,i}}{A_{n,i}} \right) = 0 \) so that

\[
\int_{\infty}^{\infty} S_{\Phi}(x) dx = 0.
\]

Moreover, in this case it follows from (2.4) and (2.6)

\[
Pr[I_{0}(X) \geq VaR_{\Phi}(X, 0)] = \begin{cases} 
Pr[I_{0}(X) \geq \chi^{-1}\Phi(\alpha)] = \rho \sum_{j=1}^{n} c_{n,j} \int_{d_{k,j}}^{\infty} S_{\Phi}(x) dx.
\end{cases}
\]

Proof of Lemma 4.3. (a) When \( S_{\Phi}^{-1}(\alpha) \leq d_{i,k} \), it follows from (2.6) that \( VaR_{\Phi}(X, 0) \leq d_{i,k} \). Thus, by (2.5) and after some algebra, it is not difficult to show that

\[
\int_{\infty}^{\infty} S_{\Phi}(x) dx = \int_{A_{n,i}}^{\infty} S_{\Phi}(x) dx = \begin{cases} 
\int_{A_{n,i}}^{\infty} S_{\Phi}(x) dx + \sum_{j=1}^{n} \int_{A_{n,i}}^{\infty} S_{\Phi} \left( \frac{x - B_{n,i}}{A_{n,j}} \right) dx
\end{cases}
\]

Furthermore, by (A.5), (4.16) and (4.17), we have for \( i = 1, \ldots, n - 1 \),

\[
\min_{d_{k,i}} \left\{ \text{CTE}_{\Phi}(d_{1,i}, \ldots, d_{n,i}, \alpha) \right\} = \begin{cases} 
\text{CTE}_{\Phi}(d_{1,i}, \ldots, d_{n,i}, \alpha)
\end{cases}
\]

Note that \( 0 < A_{n,i} < 1 \) for \( i = 1, \ldots, n - 1 \). Furthermore, substituting (2.3), (A.18) and (A.19) and the following result

\[
Pr[I_{0}(X) \geq VaR_{\Phi}(X, 0)] = \begin{cases} 
Pr[I_{0}(X) \geq \chi^{-1}\Phi(\alpha) + B_{n,i}] = \Pr[A_{n,i} X + B_{n,i} \geq A_{n,i} \chi^{-1}(\alpha) + B_{n,i}]
\end{cases}
\]

into (4.13), we obtain the required expression (4.16).
\[
= \left(1 - \frac{1}{\alpha}\right) v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^*)}{S_\alpha(x) dx < 0}. \quad (A.26)
\]

Note that the condition \( \alpha < \rho^* \) implies \( u(\rho^*) - v(\alpha) = S_\alpha^1(\rho^*) - S_\alpha^1(\alpha) \) and \( \rho^* - 1\alpha = \int_1^\infty S_\alpha(x) dx < 0. \) (A.26)

It is therefore easy to see that the minimum in (A.25) is less than any of the minimums in (A.23) and (A.24), which leads to

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \min_{d_n, \alpha} \left\{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \right\}.
\]

Hence this justifies both (4.18) and (4.19). (b) When \( \alpha = \rho^* \), we have \( d^* = S_\alpha^1(\rho^*) = S_\alpha^1(\alpha) \) and \( u(\rho^*) = g(d^*) = v(\alpha) \) so that in this case, (4.15) reduces to

\[
\text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) = v(\alpha),
\]

for any \( (d_n, 1, \ldots, d_n, n) \in D_n^0 \). Furthermore, by (A.5), (4.16) and (4.17), we have for \( i = 1, \ldots, n-1, \)

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \text{CTE}_{\alpha}(d_i, 1, \ldots, d_i, d_{i+1}, \ldots, d_n, n, \alpha)
\]

\[
= \left(1 - \frac{1}{\alpha}\right) v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^*)}{S_\alpha(x) dx < 0}, \quad (A.28)
\]

and for \( i = n, \)

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \text{CTE}_{\alpha}(d^n, 1, \ldots, d^n, \alpha)
\]

\[
= \left(1 - \frac{1}{\alpha}\right) v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^n)}{S_\alpha(x) dx < 0}, \quad (A.29)
\]

where (A.28) holds for any \( d_{i+1}, \ldots, d_n \) such that \( d^* = S_\alpha^1(\alpha) \leq d_{i+1} \leq \cdots \leq d_n \).

Thus, \( \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \) has the same minimum of \( v(\alpha) \) on all sets \( D_n^0 \), \( i = 0, 1, \ldots, n \). Hence, \( \min_{d_n} \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) = v(\alpha) \) and the minimum CTE is attained at any

\[
(d_1, \ldots, d_n) \in D_n^0 \cup \left(\frac{n-1}{i=1} \left\{ (d_1, \ldots, d_i, d_{i+1}, \ldots, d_n) : d^* \leq d_{i+1} \leq \cdots \leq d_n \right\} \right) \bigcup \{(d^*, \ldots, d^*)\}.
\]

Since \( d^* = S_\alpha^1(\alpha) \) in this case, we obtain

\[
D_n^0 \cup \left(\frac{n-1}{i=1} \left\{ (d_1, \ldots, d_i, d_{i+1}, \ldots, d_n) : d^* \leq d_{i+1} \leq \cdots \leq d_n \right\} \right) \bigcup \{(d^*, \ldots, d^*)\} = D_n^0,
\]

which consists of all \( (d_1, \ldots, d_n) \) such that \( d^* \leq d_{i+1} \leq \cdots \leq d_n \). This completes the proof of case (b).

(c) In this case, \( \alpha < \rho^* \) implies \( \frac{1}{\rho^*} - \frac{1}{\alpha} < 0 \) and \( d^* = S_\alpha^1(\rho^*) < S_\alpha^1(\alpha) \). Thus, it follows from (4.15) that

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \text{CTE}_{\alpha}(S_\alpha^1(\alpha), \ldots, S_\alpha^1(\alpha), \alpha)
\]

\[
= v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^*)}{S_\alpha(x) dx}.
\]

(A.30)

Furthermore, by (A.6), (4.16) and (4.17), we have for \( i = 1, \ldots, n-1, \)

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \text{CTE}_{\alpha}(0, \ldots, 0, S_\alpha^1(\alpha), \ldots, S_\alpha^1(\alpha), \alpha)
\]

\[
= v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^*)}{S_\alpha(x) dx}.
\]

(A.31)

and for \( i = n, \)

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \text{CTE}_{\alpha}(0, \ldots, 0, \alpha)
\]

\[
= v(\alpha) + \frac{1}{\alpha} \sum_{j=1}^{n} \frac{c_{nj} g(d^*)}{S_\alpha(x) dx}.
\]

(A.32)

Now, note that from the condition \( \alpha < S(X) \leq \rho^* \), we have

\[
\frac{1}{\alpha} \int_0^{S(X)} S_\alpha(x) dx - S_\alpha^1(\alpha) < \left(\frac{1}{\alpha}\right) S_\alpha^1(\alpha) S(X) - S_\alpha^1(\alpha) < 0
\]

and \( \frac{1}{\rho^*} - \frac{1}{\alpha} < 0 \). This leads to

\[
g(0) - v(\alpha) = (1 + \rho) E[X] - S_\alpha^1(\alpha) - \frac{1}{\alpha} \int_{S_\alpha^1(\alpha)}^{\infty} S_\alpha(x) dx
\]

\[
= \frac{1}{\rho^*} \int_0^{S_\alpha^1(\alpha)} S_\alpha(x) dx - S_\alpha^1(\alpha) + \int_{S_\alpha^1(\alpha)}^{\infty} S_\alpha(x) dx < 0.
\]

So it is easy to conclude that the minimum in (A.32) is less than any of the minimums in (A.30) and (A.31). Consequently, we have

\[
\min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} = \min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \}
\]

which confirms both (4.22) and (4.23).

(d) It is easy to verify that \( \min_{d_n} \{ \text{CTE}_{\alpha}(d_n, 1, \ldots, d_n, n, \alpha) \} \) does not exist for all other cases. \( \square \)

Lemma A.1. For any \( f(x) \in F \), \( f(x) = x - f(x) \) is increasing and concave in \( x \).
Proof. The concavity of $I_2(x)$ comes immediately from the fact that $f(x)$ is convex. Now suppose there exist two points $x_1$ and $x_2$ such that $0 \leq x_1 < x_2$ satisfying $I_2(x_1) - I_2(x_2) > 0$, i.e.,

$$f(x_2) - f(x_1) < x_2 - x_1.$$  
(A.33)

Moreover, it follows from the convexity of $f(x)$ that $f(x_1) \leq \frac{x_2 - x_1}{x_2 - x_1} f(x_1) + \frac{x_1 - x_2}{x_2 - x_1} f(x_2)$ for $x \geq x_2$, or equivalently $f(x) \geq \frac{x_2 - x_1}{x_2 - x_1} f(x_1) + \frac{x_1 - x_2}{x_2 - x_1} f(x_2)$. Hence, it follows from (A.33) that there exists a constant $\alpha_0$ such that $f(\alpha_0) > x_0$, which contradicts the assumption that $f(x) \leq x$ for all $x \geq 0$. Therefore we conclude that $I_2(x)$ is increasing. □

Lemma A.2. If $I_2(x)$ is an increasing concave function, then the distribution function of $I_2(x)$, $F_2(x)$, has at most one discontinuity on $(0, \infty)$. If such a discontinuity exists, then

$$I_2(x) = \begin{cases} g(x), & 0 < x \leq e_0, \\ g(e_0), & x > e_0, \end{cases}$$

for some strictly increasing function $g(x)$ and some constant $e_0 \in (0, \infty)$, and $g(e_0)$ is the only discontinuity of $F_2(x)$.

Proof. Since $I_2(x)$ is increasing concave, $I_2(x)$ must be either strictly increasing on $[0, \infty)$ or has the following representation of (A.34) for some strictly increasing function $g$ and some constant $e_0 \in (0, \infty)$. Otherwise, if there exists an interval $[x_1, x_2]$ such that $I_2(x)$ is constant while strictly increasing on $[x_2, x_2 + \epsilon]$ (for any real number $\epsilon > 0$), then there exists a real number $0 < \delta < \epsilon$ such that

$$\frac{\delta}{\delta + x_2 - x_1} I_2(x_1) + \frac{x_2 - x_1}{\delta + x_2 - x_1} I_2(\delta + x_2) = \frac{\delta}{\delta + x_2 - x_1} I_2(x_2) + \frac{x_2 - x_1}{\delta + x_2 - x_1} I_2(\delta + x_2),$$

which violates the concavity of the function $I_2(x)$.

If $I_2(x)$ is strictly increasing globally, by the assumption that $F_2(x)$ is a one-to-one distribution function on $(0, \infty)$, then $F_2(x)$ has no discontinuous point. On the other hand, if $I_2(x)$ takes the form of (A.34), then

$$F_2(x) = \begin{cases} F_2(0), & x = 0, \\ F_2(g^{-1}(x)), & 0 < x \leq g(e_0), \\ 1, & x > g(e_0). \end{cases}$$

Therefore $g(e_0)$ is the only discontinuous point on $(0, \infty)$. □

Proof of Lemma 4.1. We verify the result by considering two cases respectively. First of all, we suppose the distribution function of $I_2(x)$ as denoted by $F_2(x)$, is continuous at $VaR_{\alpha}(x)$. By Lemma 3.1, there exists a sequence of functions $h_n(x) \in \mathcal{H}$ such that $\lim_{n \to \infty} h_n(x) \to f(x)$ for all $x \geq 0$ and hence it follows from Proposition A.1(b) and the fact that $h_n(x)$ and $f(x)$ are increasing and continuous that

$$\lim_{n \to \infty} VaR_{h_n}(\alpha) = VaR_{f}(\alpha).$$

Consequently, by the dominated convergence theorem we obtain

$$h_n(x) - VaR_{h_n}(\alpha) \overset{\text{a.s.}}{\to} I_2(x) - VaR_{f}(\alpha),$$

which immediately leads to (4.3).

Next we suppose $VaR_{f}(\alpha)$ is one discontinuity of $F_2(x)$. By Lemma A.1, $I_2(x)$ is increasing concave in $x$, and hence it follows from Lemma A.2 that $VaR_{h_n}(\alpha)$ is the only discontinuity of $F_2(x)$, and

$$I_2(x) = \begin{cases} g(x), & 0 < x \leq e_0, \\ g(e_0), & x > e_0, \end{cases}$$

for some strictly increasing and continuous function $g(x)$ and a constant $e_0$ such that $g(e_0) = VaR_{f}(\alpha)$. Consequently

$$VaR[I_2(x) \geq VaR_{f}(\alpha)] = VaR[g(x) \geq e_0] = Pr[X \geq e_0].$$

To verify the result in this case, we adopt the technique of constructing a sequence of functions $h_n(x)$ in class $\mathcal{H}$ satisfying (4.2) and (4.3) as follows.

Denote $x_{2i+1} = i \cdot e_0/2^n$, $a_{2i} = f'_+(x_{2i+1})$, and $l_i(x) = a_{2i}^{(1)}(x - b_{2i+1}^{(1)})$ with $b_{2i} \in R$, to be a tangent curve of $f(x)$, $i = 0, 1, 2, \ldots, 2^n$. Then,

$$b_{2i+1}^{(1)} = \frac{f(x_{2i+1}) - f(x_{2i+1})}{f'_+(x_{2i+1})} = \frac{i}{2^n} e_0 - f\left(\frac{i}{2^n} e_0\right),$$

and $l_i(x) = x - g(e_0) = f(x)$, i.e. $a_{2i+1} = f'_+(x_{2i+1}) = 1$, $b_{2i} = g(e_0) = e_0 - f(e_0)$. Note that $a_{2i+1}^{(1)} < a_{2i+1}^{(2)} \leq 1$, $i = 1, 2, \ldots, 2^n$ due to the assumption that $f(x)$ is strictly increasing convex on interval $[0, e_0]$ and the fact that $0 \leq f'_+(x) \leq 1$. Take

$$h_n(x) = \max\{0, l_1(x), l_2(x), \ldots, l_{2^n-1}(x), l_{2^n}(x)\}.$$

Then for any given $\epsilon > 0$, it follows from the continuity of $f(x)$ and the construction of $h_n(x)$ that there exists a large enough integer $n$ such that $f(x) - l_i(x) < \frac{\epsilon}{2^n}$ and $h_n(x) - l_i(x) < \frac{\epsilon}{2^n}$ for $x \in [x_{2i+1}, x_{2i+2}, i = 0, 1, 2, \ldots, 2^n - 1$, and $h_n(x) = l_{2^n}(x) = f(x)$ for $x \in [e_0, \infty)$. Hence it follows

$$\lim_{n \to \infty} h_n(x) = f(x) \quad \text{for all} \quad x \geq 0.$$

Furthermore, $h_n(x)$ can be rewritten as follows.

$$h_n(x) = \sum_{i=1}^{2^n} (l_i(x) - l_{i-1}(x)) + \sum_{i=1}^{2^n} (a_{2i}^{(1)} - a_{2i-1}^{(1)}) \left(x - \frac{b_{2i} a_{2i}^{(1)} - b_{2i+1} a_{2i+1}^{(1)}}{a_{2i}^{(2)} - a_{2i-1}^{(2)}}\right) + 2^n c_{2^n}(x - d_{2^n}^{(1)}),$$

where

$$c_{2^n} = a_{2^n} - a_{2^n-1}^{(2)}, \quad d_{2^n} = b_{2^n} a_{2^n}^{(1)} - b_{2^n-1} a_{2^n-1}^{(1)} - a_{2^n}^{(1)}.$$
Now concentrate on $d_{2n,i}$, which can be expressed as follows.

\[ d_{2n,i} = i \frac{e_0}{2^n} + \frac{e_0}{2^n} f'_i \left( \frac{x+e_0}{2^n} \right) - \frac{f \left( \frac{x+e_0}{2^n} \right) - f \left( \frac{x-1}{2^n} e_0 \right)}{f'_i \left( \frac{x-1}{2^n} e_0 \right) - f'_i \left( \frac{x+e_0}{2^n} \right)}. \]  

By the convexity of $f(x)$ we have

\[ d_{2n,i} - d_{2n,i-1} = e_0 \frac{e_0}{2^n} \left[ f'_i \left( \frac{x+e_0}{2^n} \right) - f \left( \frac{x+e_0}{2^n} \right) \right] + \frac{e_0}{2^n} \left[ f'_i \left( \frac{x+e_0}{2^n} \right) - f'_i \left( \frac{x-1}{2^n} e_0 \right) \right] \]
\[ = \frac{e_0}{2^n} \left[ f'_i \left( \frac{x+e_0}{2^n} \right) - f'_i \left( \frac{x-1}{2^n} e_0 \right) \right] > 0, \]

which means $d_{2n,i}$ is increasing in $i$ and hence $d_{2n,2^n}$ is the maximum among $\{d_{2n,i}, i = 1, 2, \ldots, n\}$. As a result, we know that $h_{b_i}$ can be represented in the form of (A.35) by replacing $e_0$ with $d_{2n,2^n}$, i.e.,

\[ h_{b_i}(x) = \begin{cases} k(x), & 0 < x \leq d_{2n,2^n}, \\ k(d_{2n,2^n}), & x > d_{2n,2^n}, \end{cases} \]  

(A.37)

for some strictly increasing and continuous function $k(x)$. Moreover, it follows from (A.36) and the convexity of $f(x)$ that

\[ d_{2n,2^n} = e_0 + \frac{e_0}{2^n} f'_i \left( \frac{2^n-1}{2^n} e_0 \right) - \frac{f \left( e_0 \right) - f \left( \frac{2^n-1}{2^n} e_0 \right)}{1 - f'_i \left( \frac{2^n-1}{2^n} e_0 \right)} \leq e_0, \]  

(A.38)

and

\[ d_{2n,2^n} = e_0 + \frac{e_0}{2^n} f'_i \left( \frac{2^n-1}{2^n} e_0 \right) - \frac{f \left( e_0 \right) - f \left( \frac{2^n-1}{2^n} e_0 \right)}{1 - f'_i \left( \frac{2^n-1}{2^n} e_0 \right)} \]
\[ = e_0 - \frac{e_0}{2^n} \left[ 1 - f'_i \left( \frac{2^n-1}{2^n} e_0 \right) \right] + \frac{e_0}{2^n} f'_i \left( e_0 \right) - f \left( e_0 \right) - f \left( \frac{2^n-1}{2^n} e_0 \right) \]
\[ \geq e_0 - \frac{e_0}{2^n}. \]  

(A.39)

Combining (A.38) and (A.39) immediately yields

\[ \lim_{n \to \infty} d_{2n,2^n} = e_0. \]

Now turn to investigate $Pr(h_{b_i}(X) \geq \text{VaR}_{\alpha}(X))$. From (A.37) and (A.38) we have

\[ Pr(h_{b_i}(X) \geq \text{VaR}_{\alpha}(X)) = Pr(X \geq d_{2n,2^n}) \geq Pr[X \geq e_0] \geq \alpha, \]

and

\[ Pr(h_{b_i}(X) > \text{VaR}_{\alpha}(X)) = 0. \]

These results in turn imply $\text{VaR}_{\alpha}(X)(\alpha) = h_{b_i}(d_{2n,2^n})$, and consequently,

\[ \lim_{n \to \infty} Pr(h_{b_i}(X) \geq \text{VaR}_{\alpha}(X)) = \lim_{n \to \infty} Pr(h_{b_i}(X) \geq \text{VaR}_{\alpha}(X)) \]
\[ = \lim_{n \to \infty} Pr[X \geq d_{2n,2^n}] \]
\[ = Pr[X \geq e_0] = Pr[I_{\alpha}(X) \geq \text{VaR}_{\alpha}(X)]. \]

This completes the proof. \( \square \)

References


