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Test of fit for a Laplace distribution against heavier tailed alternatives

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Abstract

Over the last decade there has been a marked interest to a Laplace distribution and its properties and generalizations, especially in a framework of financial applications. Such an interest has led to a revision and discussion of available goodness-of-fit procedures for a Laplace distribution. Indeed, since most of the studies which employ the Laplace distribution are concerned with modelling heavy tailed patterns, the modern class of possible alternatives is way broader than just testing the Laplace vs. normal distribution. In this paper we propose a new test of fit for a Laplace distribution against deviations with heavier tails than that of the reference Laplace distribution. The proposed goodness-of-fit procedure is based on sample skewness and kurtosis and a robust L_1 -estimator of scale about a sample median. The developed test statistic is shown to asymptotically follow a χ^2 -distribution with two degrees of freedom. Performance of the new goodness-of-fit test is illustrated by simulations and a case study.

Keyword: Goodness-of-fit tests; Test for a Laplace distribution; Robust estimators of scale; Skewness; Kurtosis; moment-based procedures of fit.

1 Introduction

Recently, there has been an increasing interest in properties and generalization of a Laplace distribution (see, for example, Puig and Stephens, 2000; Kotz et al., 2001; Kozubowski and Podgorski, 2001; Haas et al., 2006; Srivastava et al., 2006). A Laplace (also known as double exponential) distribution belongs to a family of geometric stable distributions and possesses a number of favorable characteristics which make it attractive for many applications. In particular, a simple closed form, stability and robustness to model misspecification of a Laplace distribution are found to be especially appealing in modelling heavy tailed processes in finance, engineering, astronomy and environmental sciences (Granger and Ding, 1995; Mitnik et al., 1998; Linden, 2001; Kotz et al., 2001; Gang et al, 2003; Sabarinath and Anilkumar, 2008, and references therein). Thus, the question is raised on how well the observed data fit the Laplace distribution. Moreover, since most of the studies which employ the Laplace distribution are concerned with modelling heavy tailed patterns, the class of possible alternatives is way broader than just testing the Laplace vs. normal distribution (Heyde and Kou, 2004; Glynn and Torres, 1996; Haas et al., 2006).

Most of currently employed goodness-of-fit tests for a Laplace distribution are based on estimation of an empirical distribution function (EDF). Among such tests of fit are the Cramer-von Mises W^2 , the Watson U^2 , the Anderson-Darling A^2 , the Kolmogorov-Smirnov D and the Kuiper V (see Puig and Stephens, 2000 and Kotz et al., 2001, for an overview). As an alternative to the EDF type of tests, in this paper we propose a new goodness-of-fit test for a Laplace distribution, based on the robust L_1 estimates of skewness and kurtosis. Recently Best et al. (2008) suggest a smooth test of fit for a Laplace distribution which is related to the sample

estimates of skewness and kurtosis where the classical moment estimators are utilized. Since the Maximum Likelihood (ML) sample estimator of scale for a Laplace distribution is yielded by an L_1 deviation about a sample median, it seems natural to utilize this robust estimator in a goodness-of-fit procedure rather than a classical sample standard deviation. We show that the new test statistic is asymptotically χ^2 -distributed with two degrees of freedom. As indicated by simulation studies, the new L_1 -based test of fit achieves high performance in detecting symmetric and moderately skewed heavier tailed alternatives to the Laplace distribution and can be recommended over other goodness-of-fit tests for such departures from the Laplace distribution.

The paper proceeds as follows. In section 2 we discuss the new test statistic and its asymptotic properties. In section 3 we present a simulation study on the size and power of the new goodness-of-fit test and illustrate the new procedure by application to returns of the German stock index (DAX). Section 4 summarizes the main results and concludes the paper.

2 A new goodness-of-fit test statistic and its asymptotic properties

The *classical* Laplace distribution $\mathcal{CL}(\alpha, \beta)$, also known as double exponential, is a probability distribution with density function

$$f(x; \alpha, \beta) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}, \quad \alpha \in R, \beta > 0, \quad (1)$$

and distribution function

$$F_0(x, \alpha, \beta) = \begin{cases} \frac{1}{2} e^{-|x-\alpha|/\beta}, & \text{if } x \leq \alpha, \\ 1 - \frac{1}{2} e^{-|x-\alpha|/\beta}, & \text{if } x \geq \alpha. \end{cases} \quad (2)$$

In particular, mean and variance of (1) are given by α and $2\beta^2$ respectively (see Kotz et al., 2001; Balakrishnan and Nevzorov, 2003, and references therein).

If we re-parameterize x by $t = (x - \alpha)/\beta$, we obtain a *standard classical* Laplace distribution $\mathcal{CL}(0, 1)$ with density

$$f(t; 0, 1) = \frac{1}{2} e^{-|t|}. \quad (3)$$

Location and scale parameters α and β are generally unknown and estimated from the observed sample X_1, X_2, \dots, X_n . In particular, we can utilize the Maximum Likelihood (ML) sample estimates, $\hat{\alpha}_n$ and $\hat{\beta}_n$, where $\hat{\alpha}_n$ is a sample median and $\hat{\beta}_n$ is an average absolute deviation from $\hat{\alpha}_n$, i.e.

$$\hat{\beta}_n = \frac{1}{n} \sum_{i=1}^n |X_i - \hat{\alpha}_n|. \quad (4)$$

Note that the ML estimate of population standard deviation is then given by

$$\zeta_n = \sqrt{2}\hat{\beta}_n = \frac{\sqrt{2}}{n} \sum_{i=1}^n |X_i - \hat{\alpha}_n|. \quad (5)$$

Most of goodness-of-fit tests for the Laplace distribution are based on estimation of the empirical distribution function (EDF) F_n , defined by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x), \quad (6)$$

where I is the indicator function; and then by comparing F_n with the hypothesized distribution F_0 (Yen and Moore, 1988; Puig and Stephens, 2000; Kotz et al., 2001). The most popular EDF based test statistics are the Cramer-von Mises W^2 , the Watson U^2 , the Anderson-Darling A^2 , the Kolmogorov-Smirnov D and the Kuiper V (for more detailed discussion and asymptotic results on the EDF tests see Puig and Stephens, 2000). The EDF test tabulated critical values for different sample sizes can be found, for example, in Yen and Moore (1988), Puig and Stephens (2000), Kotz et al. (2001) and Chen (2002).

In this paper we propose an alternative goodness-of-fit test statistic for the Laplace distribution, based on the sample skewness and kurtosis¹. Let X_1, X_2, \dots, X_n be a sample of independent and identically distributed (i.i.d.) random variables from a Laplace distribution $\mathcal{CL}(\alpha, \beta)$. Define measures of population skewness and kurtosis as

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = 0, \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 6. \quad (7)$$

Since the ML estimator of scale for the Laplace distribution is yielded by (5), we can utilize (5) rather than a classical sample standard deviation in denominators of (7). Hence, our new estimates of sample skewness and kurtosis take the form:

$$\begin{aligned} \sqrt{v_1} &= \frac{n^{-1} \sum_{k=1}^n (X_k - \bar{X}_n)^3}{\zeta_n^3}, \\ v_2 &= \frac{n^{-1} \sum_{k=1}^n (X_k - \bar{X}_n)^4}{\zeta_n^4}, \end{aligned} \quad (8)$$

where \bar{X} is a sample mean.

Now we can use $\sqrt{v_1}$ and v_2 to construct a test statistic for a Laplace distribution, which takes the form

$$K = \frac{n}{C_1} (\sqrt{v_1})^2 + \frac{n}{C_2} (v_2 - \beta_2)^2, \quad C_1, C_2 > 0. \quad (9)$$

¹Moment-based procedures such as the D'Agostino-Pearson, the Bowman-Shenton and the Jarque-Bera tests of fit for a normality are widely accepted in statistics, econometrics and other fields of study (D'Agostino and Pearson, 1973; Bowman and Shenton, 1975; Jarque and Bera, 1980; Thadewald and Buning, 2007 and references therein). Bowman and Shenton (1986) propose to employ the moment-based procedure to assess a number of non-normal distributions but the analysis does not include a Laplace distribution.

As follows from the next theorem, the new test statistic K is asymptotically χ^2 -distributed with two degrees of freedom.

Theorem Let X_1, X_2, \dots, X_n be a sample of i.i.d. observations from $\mathcal{CL}(\alpha, \beta)$. Then $\sqrt{v_1}$ and v_2 given by (8) are consistent estimates of population skewness $\sqrt{\beta_1}$ and kurtosis β_2 respectively. In addition, $\sqrt{v_1}$ and v_2 are asymptotically normally distributed and independent,

$$\sqrt{n} \begin{bmatrix} \sqrt{v_1} \\ v_2 - 6 \end{bmatrix} \Rightarrow N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \right), \quad C_1, C_2 > 0. \quad (10)$$

Proof. The result of Gastwirth (1982) implies that for $\mathcal{CL}(\alpha, \beta)$, ζ_n is a consistent asymptotically normally distributed estimate of the population standard deviation $\sigma = \sqrt{2}\beta$, i.e.

$$\lim_{n \rightarrow \infty} E\zeta_n = \sigma \quad (11)$$

and

$$\sqrt{n}(\zeta_n - \sigma) \sim N \left(0, \sigma \sqrt{\sqrt{2} - 1} \right). \quad (12)$$

Hence, the theorem on functions of converging random variables (Lehmann, 2004) implies that

$$\begin{aligned} \sqrt{v_1} &\xrightarrow{P} \sqrt{\beta_1}, \\ v_2 &\xrightarrow{P} \beta_2, \end{aligned} \quad (13)$$

i.e. $\sqrt{v_1}$ and v_2 are asymptotically consistent estimators of the population skewness $\sqrt{\beta_1}$ and kurtosis β_2 respectively.

Finally, since a Laplace distribution is symmetric and possesses population moments of any order, the asymptotic normality and independence of $\sqrt{v_1}$ and v_2 follows from (13) and the multivariate Slutsky theorem (Lehmann, 2004).

In view of (10), the test statistic K asymptotically follows a χ^2 -distribution with two degrees of freedom, i.e. $K \sim \chi_2^2$. Therefore, the implied one-sided rejection region is

$$\text{reject } H_0 : \mathcal{CL}(\alpha, \beta), \text{ if } K \geq \chi_{1-\alpha, 2}^2, \quad (14)$$

where $\chi_{1-\alpha, 2}^2$ is the upper α -percentile of the χ_2 -distribution with 2 degrees of freedom.

Remark 1. Constants C_1 and C_2 can be obtained using the multivariate Taylor-expansions or the Pade approximations (Bowman and Shenton, 1986). However, these calculations are very cumbersome and impractical for many applications. Instead we can select C_1 and C_2 for a fixed nominal level α using the Monte Carlo

simulations. In particular, we propose to utilize $C_1 = 60$ and $C_2 = 1200$ for the nominal test level $\alpha = 0.05$ for small and moderate sample sizes. Notice that choice of constants C_1 and C_2 plays a role only if one desires to use the χ_2^2 -approximation for critical values. On contrary, if the exact (Monte Carlo simulated) critical values are utilized instead, choice of C_1 and C_2 does not affect size and power of the test anymore. It turns out that, similarly to the case of a normal distribution (Thadewald and Buning, 2007), sample kurtosis v_2 for a Laplace distribution converges very slowly to its asymptotic distribution. Hence, the exact (simulated) critical values for K are preferred.

Remark 2. Occasionally we refer to the test statistic K as the *robust* $(\sqrt{v_1}, v_2)$ -type statistic. Here the term *robust* relates to the fact that we utilize an L_1 -deviation about a sample median (5) in the denominators of skewness and kurtosis. As noted by Croux et al. (2006), using robust estimators in goodness-of-fit statistics can sometimes lead to power increase of a related test.

Remark 3. Since a sample median $\hat{\alpha}_n$ is a ML estimator of location for a Laplace distribution, we can consider sample estimators of the third and fourth moments about median $\hat{\alpha}_n$ rather than about a sample mean \bar{X} , which leads to a test statistic

$$K' = \frac{n}{C_1'} \left[\frac{n^{-1} \sum_{k=1}^n (X_k - \bar{X}_n)^3}{\zeta_n^3} \right]^2 + \frac{n}{C_2'} \left[\frac{n^{-1} \sum_{k=1}^n (X_k - \bar{X}_n)^4}{\zeta_n^4} - 6 \right]^2, \quad C_1, C_2 > 0.$$

Our simulation study indicates that any increase in power yielded by K' when compared to K , is relatively incremental if any. Hence, we exclude K' from further discussion.

Remark 4. Note that instead of ζ_n it is possible to use other consistent robust estimators of scales, e.g. Hubert's median absolute deviation (MAD) (Hall and Welsh, 1985). However, if the estimator of scale is too robust and, thus, very insensitive to deviations in the tails, it can lead to a low power of the test in detecting alternative distributions.

3 Size and Power Studies

In this section we assess size and power of the test statistic K for a Laplace distribution $\mathcal{CL}(0, 1)$. Our Monte Carlo simulation study on size of the test (see Table 1) indicates that the χ_2^2 -parametric approximation of critical values for K is inappropriate in small to moderately large samples, due to very slow convergence of a sample kurtosis v_2 . Since critical values for other available tests for a Laplace distribution can only be found either from a special tables or using Monte Carlo simulations (see Yen and Moore (1988), Puig and Stephens (2000), Kotz et al. (2001) and Chen (2002)), a strong preference to employ empirical critical values for K for small and moderate samples cannot be considered as a disadvantage for the new goodness-of-fit test².

²The R function for the new goodness-of-fit test, with options for empirical and approximated critical values, is to be available from the R package *lawstat*.

Table 1: Size of the test statistic K when the χ_2^2 -approximated critical values are utilized, i.e. the number of rejection of the null hypothesis when the actual data follow $\mathcal{CL}(0, 1)$. The nominal level $\alpha = 0.05$. Number of Monte Carlo simulations is 100,000. Chosen constants are $C_1 = 60$ and $C_2 = 1200$.

| $\alpha = 0.05$ | | | | |
|-----------------|----------|----------|----------|-----------|
| | $n = 40$ | $n = 60$ | $n = 80$ | $n = 100$ |
| R | 0.0400 | 0.0443 | 0.0470 | 0.0539 |

To assess power of the new goodness-of-fit test, we apply the statistic K to a number of symmetric and asymmetric alternatives such as normal, logistic, exponential, Cauchy, t - and Tukey's contaminated normal distributions³ that are commonly utilized in other studies on a Laplace distribution (Puig and Stephens, 2000; Best et al. 2008, etc). In addition, since the main focus of the new test statistic K is to identify heavier tailed deviations from a Laplace distribution, we also consider a Normal Inverse Gaussian (NIG) distribution. The NIG distribution is completely determined by the four parameters, α, β, μ and $\delta \in R$, and the appropriate tuning of these four parameters enables to obtain a broad range of continuous distributions of different shapes (Atkinson, 1982; Barndorff-Nielsen and Blaesild, 1983)⁴. Due to its flexible closed form makes, NIG is very popular in a variety of applications and, in particular, is widely used for modelling heavy-tailed financial processes (for more on applications of NIG distributions see, for example, Barndorff-Nielsen, 1997, and references therein).

We present power study of six goodness-of-fit tests for the Laplace distribution. In particular, we compare the new test statistic K with the Cramer-von Mises W^2 , the Watson U^2 , the Anderson-Darling A^2 , the Kolmogorov-Smirnov D and the Kuiper V . Statistics W^2, U^2 and A^2 belong to the Cramer-von Mises family while D and V belong to the Kolmogorov-Smirnov family of tests (see Puig and Stephens, 2000). (As shown by Best et al. (2008), the smooth test of Best et al. (2008), the maximum entropy test of Choi and Kim (2006) and the Empirical Characteristic Function (ECF) based test of Meintanis (2004) provide incremental if any increase in power comparing to the Anderson-Darling A^2 test for detecting heavier tailed alternatives to a Laplace distribution. Since such heavier tailed deviations are of the main interest of this paper, we omit the smooth, the maximum entropy and ECF-based tests from our simulation study.) To ensure that the correct size of the test is preserved, only the exact (simulated) critical values are utilized for the power analysis. The first row in Table 3 corresponds to the actually observed size

³The Tukey family of contaminated normal distributions has a cdf $F_{\text{CN}_{\lambda, \alpha}}(x) = (1 - \alpha)\Phi(x) + \alpha\Phi(x/\lambda)$ and the respective pdf $f_{\text{CN}_{\lambda, \alpha}} = (1 - \alpha)\phi(x) + (\alpha/\lambda)\phi(x/\lambda)$, where Φ and ϕ are cdf and pdf of a standard normal distribution $N(0, 1)$, $0 \leq \alpha \leq 1$ and $\lambda > 0$ are constants. For more details and discussions on Tukey's contaminated normal distribution see Tukey (1960), Johnson and Kotz (1970) AND Gleason (1993).

⁴For a NIG distribution, α, β, μ and δ are shape, skewness, location and scale parameters respectively. The mean, variance, skewness and kurtosis of NIG are defined respectively by $\mu + \beta\delta/\gamma$, $\delta\alpha^2/\gamma^3$, $3\beta/\alpha\sqrt{\delta\gamma}$ and $3(1 + 4\beta^2/\alpha^2)/\delta\gamma$, where $\gamma = \sqrt{\alpha^2 - \beta^2}$.

of the test based on the selected critical values.

For symmetric alternatives with tails heavier than $\mathcal{CL}(0, 1)$ such as Tukey's contaminated normal $CN_{3.2,0.2}$ and $CN_{3.5,0.1}$, NIG_1 , NIG_2 , Cauchy and t -distribution with three degrees of freedom, the new test statistic K shows the best performance, especially for small and moderate sample sizes, followed by the Watson U^2 and the Kuiper V . All six tests show relatively low power in detecting a t -distribution with four degrees of freedom. As expected, for symmetric deviations with lighter than $\mathcal{CL}(0, 1)$ tails such as normal and logistic distributions, the highest power is provided by the Watson U^2 and the Kuiper V , while the lowest power is yielded by K .

For skewed distribution with moderately heavy tails such as exponential, the best result is shown by A^2 and the worst result is provided by K ; the remaining statistics perform similarly. However, when the degree of heavy tailness increases as in NIG_3 and NIG_4 , the new test statistic K noticeably gains power and shows the best performance, followed by the Anderson-Darling A^2 and the Watson U^2 .

Overall, we conclude that the new test statistic K performs competitively in detecting symmetric and moderately skewed deviations with profound heavy tails, i.e. heavier than the reference Laplace distribution, especially in small and moderate sample sizes, which is of particular interest in financial applications.

Table 2: Power study for the Laplace distribution, $\alpha = 0.05$. Number of MC simulations is 100,000.

| | $\sqrt{\beta_1}$ | β_2 | n | W^2 | U^2 | A^2 | D | V | K |
|--|------------------|-----------|-----|-------|-------|-------|------|------|------|
| Laplace | 0.0 | 6.0 | 20 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | | 50 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | | 100 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $N(0, 1)$ | 0.0 | 3.0 | 20 | 0.08 | 0.11 | 0.07 | 0.09 | 0.10 | 0.00 |
| | | | 50 | 0.18 | 0.33 | 0.15 | 0.20 | 0.29 | 0.00 |
| | | | 100 | 0.37 | 0.68 | 0.35 | 0.37 | 0.59 | 0.00 |
| logistic | 0.0 | 4.2 | 20 | 0.06 | 0.07 | 0.05 | 0.06 | 0.07 | 0.02 |
| | | | 50 | 0.10 | 0.17 | 0.09 | 0.12 | 0.16 | 0.01 |
| | | | 100 | 0.17 | 0.35 | 0.14 | 0.19 | 0.31 | 0.01 |
| Tukey $CN_{3,2,0.2}$ | 0.0 | 10.1 | 20 | 0.07 | 0.13 | 0.09 | 0.08 | 0.12 | 0.24 |
| | | | 50 | 0.11 | 0.20 | 0.15 | 0.13 | 0.19 | 0.35 |
| | | | 100 | 0.17 | 0.35 | 0.18 | 0.17 | 0.30 | 0.38 |
| NIG ₁ $\alpha = 0.4, \beta = 0$ $\mu = 1, \delta = 0.6$ | 0.0 | 12.5 | 20 | 0.07 | 0.11 | 0.09 | 0.08 | 0.09 | 0.19 |
| | | | 50 | 0.10 | 0.18 | 0.12 | 0.11 | 0.16 | 0.32 |
| | | | 100 | 0.15 | 0.27 | 0.18 | 0.16 | 0.24 | 0.46 |
| Tukey $CN_{3,5,0.1}$ | 0.0 | 13.7 | 20 | 0.07 | 0.12 | 0.08 | 0.08 | 0.11 | 0.26 |
| | | | 50 | 0.10 | 0.18 | 0.13 | 0.11 | 0.16 | 0.43 |
| | | | 100 | 0.17 | 0.34 | 0.19 | 0.17 | 0.29 | 0.45 |
| NIG ₂ $\alpha = 0.7, \beta = 0$ $\mu = 1, \delta = 0.2$ | 0.0 | 21.4 | 20 | 0.12 | 0.18 | 0.14 | 0.12 | 0.16 | 0.29 |
| | | | 50 | 0.21 | 0.35 | 0.23 | 0.21 | 0.32 | 0.50 |
| | | | 100 | 0.38 | 0.57 | 0.41 | 0.34 | 0.52 | 0.70 |
| t_3 | - | - | 20 | 0.07 | 0.08 | 0.07 | 0.07 | 0.07 | 0.13 |
| | | | 50 | 0.08 | 0.11 | 0.08 | 0.08 | 0.11 | 0.21 |
| | | | 100 | 0.09 | 0.14 | 0.10 | 0.10 | 0.13 | 0.31 |
| Cauchy | - | - | 20 | 0.53 | 0.62 | 0.55 | 0.50 | 0.60 | 0.68 |
| | | | 50 | 0.86 | 0.92 | 0.86 | 0.83 | 0.90 | 0.94 |
| | | | 100 | 0.98 | 0.99 | 0.99 | 0.98 | 0.99 | 0.99 |
| exp | 2.0 | 9.0 | 20 | 0.43 | 0.44 | 0.54 | 0.47 | 0.44 | 0.15 |
| | | | 50 | 0.88 | 0.93 | 0.97 | 0.96 | 0.96 | 0.25 |
| | | | 100 | 0.99 | 0.99 | 1.00 | 0.99 | 0.99 | 0.41 |
| NIG ₃ $\alpha = 1, \beta = 0.5$ $\mu = 1, \delta = 0.43$ | 2.5 | 16.1 | 20 | 0.12 | 0.13 | 0.16 | 0.11 | 0.11 | 0.21 |
| | | | 50 | 0.23 | 0.24 | 0.32 | 0.21 | 0.21 | 0.37 |
| | | | 100 | 0.41 | 0.40 | 0.58 | 0.38 | 0.35 | 0.58 |
| NIG ₄ $\alpha = 0.5, \beta = 0.2$ $\mu = 1, \delta = 0.5$ | 2.5 | 21.5 | 20 | 0.12 | 0.15 | 0.15 | 0.12 | 0.14 | 0.25 |
| | | | 50 | 0.21 | 0.28 | 0.27 | 0.21 | 0.26 | 0.44 |
| | | | 100 | 0.37 | 0.46 | 0.48 | 0.35 | 0.40 | 0.63 |

Example. Now let us apply all six goodness-of-fit tests to 100 returns of closing prices of the German Stock Index (DAX). The data are observed daily from January 1, 1991, excluding weekends and public holidays. We find that the Cramer-von Mises W^2 and the Kolmogorov-Smirnov D tests do not reject the null hypothesis of a Laplace distribution and provide respectively p -values of 0.17 and 0.12. The Anderson-Darling A^2 statistic yields a significant p -value of 0.07; the Watson U^2 test rejects the null hypothesis with a statistically significant p -value of 0.032. On contrary, both the new K and the Kuiper V tests yield highly significant p -values of 10^{-12} and 0.007 respectively. Such conclusions are consistent with the results of Haas et al. (2006) who also find that the Laplace distribution is inadequate for the German stocks.

4 Conclusion

In this paper we propose a new moment-based goodness-of-fit test for a Laplace distribution. The idea of the new test is to utilize a Maximum Likelihood (ML) estimator of scale in the denominators of skewness and kurtosis. In particular, such ML estimator of spread for a Laplace distribution is given by an L_1 -deviation about a sample median which is typically found to be more resistant to outliers. Remarkably, robust estimators in tests of fit can frequently lead to power increase of a related procedure (Croux et al., 2006; Gel and Gastwirth, 2008). Indeed, when compared with other goodness-of-fit tests, the new statistic is shown to achieve higher power in assessing a broad range of symmetric or moderately skewed alternatives with heavier tails than that of the reference Laplace distribution and, thus, can be of particular use in financial data modelling. The new test statistic is easy to compute and asymptotically follows a χ^2 -distribution with two degrees of freedom. In our future research, we plan to investigate employment of various robust moment-based procedures for a Laplace distribution as well as their application to goodness-of-fit testing for other location-scale distributions.

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