Short Course

Theory and Practice of Risk Measurement

Part 4

Selected Topics and Recent Developments on Risk Measures

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Part 4

- Risk sharing
- Regulatory arbitrage
- Elicitability and convex level set
- Change of currency
- Robustness
- Summary

Risk Sharing

General setup

- n agents sharing a total risk (or asset) $X \in \mathcal{X}$
- $\rho_1, \ldots, \rho_n$: underlying risk measures

Target: for $X \in \mathcal{X}$,

$$\text{minimize} \quad \sum_{i=1}^{n} \rho_i(X_i) \quad \text{subject to } X_1 + \cdots + X_n = X, \qquad (1)$$

and find an optimal allocation of X: a solution to (1) (if it exists)

- We consider arbitrary allocations

Risk Sharing

Some interpretations

- Regulatory capital reduction within a single firm
- Regulatory capital reduction for a group of firms
- Insurance-reinsurance contracts and risk-transfer
- Risk redistribution among agents

Risk Sharing

Some classic references in the mathematical finance and insurance literature

- Barrieu-El Karoui (2005 FS)
- Jouini-Schachermayer-Touzi (2008 MF)
- Filipovic-Svindland (2008 FS)
- Cui-Yang-Wu (2013 IME)
- Delbaen (2012)

Inf-convolution

The set of allocations of $X \in \mathcal{X}$:

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

The inf-convolution of n risk measures is a functional $\square_{i=1}^n \rho_i$ mapping $\mathcal{X}$ to $[-\infty, \infty]$:

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \ldots, X_n) \in \mathbb{A}_n(X) \right\}.$$

Optimal Allocations

Definition

For monetary risk measures $\rho_1, \ldots, \rho_n$,

(i) $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is an optimal allocation if $\sum_{i=1}^n \rho_i(X_i) \leq \sum_{i=1}^n \rho_i(Y_i)$ for all $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$.

(ii) $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is a Pareto-optimal allocation if for all $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i(Y_i) \leq \rho_i(X_i)$ for all $i = 1, \ldots, n$ implies that $\rho_i(Y_i) = \rho_i(X_i)$ for all $i = 1, \ldots, n$.

Obviously, an allocation $(X_1^*, \ldots, X_n^*)$ of X is optimal if and only if

$$\sum_{i=1}^n \rho_i(X_i^*) = \mathop{\square}_{i=1}^n \rho_i(X).$$

Proposition (*)

For monetary risk measures, an allocation is optimal if and only if it is Pareto-optimal.

Inf-convolution

Proposition (*)

Suppose that $\rho_1, \ldots, \rho_n$ *are monetary risk measures and* $\Box_{i=1}^n \rho_i > -\infty$ *on* $\mathcal{X}$.

(i) $\Box_{i=1}^n \rho_i$ *is a monetary risk measure.*

(ii) *If* $\rho_1, \ldots, \rho_n$ *are convex, then* $\Box_{i=1}^n \rho_i$ *is a convex risk measure.*

(iii) *If* $\rho_1, \ldots, \rho_n$ *are coherent, then* $\Box_{i=1}^n \rho_i$ *is a coherent risk measure.*

Inf-convolution

Theorem (*)

For monetary risk measures $\rho_1, \dots, \rho_n$ *with respective acceptance set* $\mathcal{A}_1, \dots, \mathcal{A}_n$, *the acceptance set of* $\square_{i=1}^n \rho_i$ *is* $\sum_{i=1}^n \mathcal{A}_i$.

Theorem (Barrieu-El Karoui 2005 FS*)

For convex risk measures $\rho_1, \dots, \rho_n$ *with respective minimum penalty functions* $\alpha_1, \dots, \alpha_n$, *the minimum penalty function of* $\square_{i=1}^n \rho_i$ *is* $\sum_{i=1}^n \alpha_i$.

Regulatory Arbitrage

A firm may have an incentive to split its total business into n subsidies to reduce its regulatory capital

- Write $X = \sum_{i=1}^{n} X_i$ and measure each X_i with ρ
- Compare $\rho(X)$ and $\sum_{i=1}^{n} \rho(X_i)$
- Make $\sum_{i=1}^{n} \rho(X_i)$ small
- Regulatory arbitrage: $\rho(X) - \sum_{i=1}^{n} \rho(X_i)$

Example of VaR

An example of VaR_p, $p \in (0,1)$: for any risk $X > 0$ and $n > 1/(1-p)$, we can build

$$X_i = X \mathrm{I}_{A_i}, \quad i = 1, \cdots, n$$

where $\{A_i, \ i = 1, \ldots, n\}$ is a partition of Ω and $\mathrm{VaR}_p(A_i) < 1 - p$. Then $\mathrm{VaR}_p(X_i) = 0$. Therefore

$$\sum_{i=1}^{n} X_i = X$$

and

$$\sum_{i=1}^{n} \mathrm{VaR}_p(X_i) = 0.$$

Regulatory Arbitrage

Define, for $X \in \mathcal{X}$,

$$\Psi_\rho(X) = \inf\left\{\sum_{i=1}^{n} \rho(X_i) : n \in \mathbb{N},\ (X_1, \ldots, X_n) \in \mathbb{A}_n(X)\right\}.$$

- $\Psi_\rho(X)$ is the least amount of capital requirement according to ρ if the risk X can be divided arbitrarily.
- $\Psi_\rho \le \rho$.
- $\Psi_\rho = \rho$ if and only if ρ is subadditive.
- Regulatory arbitrage of ρ: $\Phi_\rho(X) = \rho(X) - \Psi_\rho(X)$.

Regulatory Arbitrage

We may categorize risk measures into four cases:

Definition (Wang, 2016 QF)

A risk measure ρ is

(i) free of regulatory arbitrage if $\Phi_\rho(X) = 0$ for all $X \in \mathcal{X}$,

(ii) of limited regulatory arbitrage if $\Phi_\rho(X) < \infty$ for all $X \in \mathcal{X}$,

(iii) of unlimited regulatory arbitrage if $\Phi_\rho(X) = \infty$ for some $X \in \mathcal{X}$,

(iv) of infinite regulatory arbitrage if $\Phi_\rho(X) = \infty$ for all $X \in \mathcal{X}$.

Regulatory Arbitrage of VaR

Theorem: Wang, 2016 QF

For $p \in (0,1)$, VaR_p is of infinite regulatory arbitrage. That is, $\Phi_{\mathrm{VaR}_p}(X) = \infty$ for all $X \in \mathcal{X}$.

- VaR is vulnerable to manipulation of risks.

Regulatory Arbitrage of General Risk Measures

Theorem: Wang, 2016 QF

The following hold:

(i) If ρ is a distortion risk measure, then ρ is of limited regulatory arbitrage if and only if $\rho(X) \geq \mathbb{E}[X]$ for all $X \in \mathcal{X}$.

(ii) If ρ is a law-determined convex risk measure, then ρ is of limited regulatory arbitrage.

In either case, Ψ_ρ is a coherent risk measure; thus, ρ is free of regulatory arbitrage if and only if it is coherent.

- In either case, Ψ_ρ is the largest coherent risk measure dominated by ρ.

Backtesting

Recall from R1, Page 41, Question 8

"… *robust backtesting* …"

Backtesting

(i) estimate a risk measure from past observations;

(ii) test whether (i) is appropriate using future observations;

(iii) purpose: monitor, test or update risk measure forecasts;

(iv) particularly relevant for market risk (daily forecasts).

For VaR, a simple procedure is available.

VaR backtesting:

Suppose that you have iid risks X_t, $t \geq 0$;

(1) suppose the estimated/modeled $\mathrm{VaR}_p(X_{t+1})$ is V_{t+1} at time t;

(2) consider random variables $A_t = \mathrm{I}_{\{X_t > V_t\}}$, $t > 0$;

(3) standard hypothesis testing methods for H_0: A_t are iid Bernoulli$(1-p)$ random variables.

For ES, a simple and intuitive procedure does not exist. Why?

Backtesting

- Not all risk measures can be backtested, and it is not easy to say which ones can
- VaR: just test whether losses exceed VaR_p $p\%$ of the times (model independent). Such good property is rare for risk measures.
- ES: backtesting procedures are model dependent
- Mode: probably impossible to backtest

Elicitability

- In 2011, a notion is proposed for comparing risk measure forecasts: elicitability, Gneiting (2011, JASA).

Quoting Acerbi and Szekely (2014 Risk):

"Eliciwhat?"

Risk professionals had never heard of elicitability until 2011, when Gneiting proved that ES is not elicitable as opposed to VaR. This result sparked a confusing debate.

Elicitability

- Roughly speaking, a law-determined risk measure (statistical functional) is elicitable if ρ is the unique solution to the following equation:

$$\rho(X) = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \mathbb{E}[s(x, X)], \ X \in \mathcal{X}$$

  where

  - $s : \mathbb{R}^2 \to [0, \infty)$ is a strictly consistent scoring function (that is, $s(x, y) = 0$ if and only if $x = y$);
  - clearly, elicitability requires $\rho(c) = c, \ c \in \mathbb{R}$ (standardization); in the following, we always assume this.

Elicitability

Examples (assuming all integrals are finite):

- the mean is elicitable with
$$s(x, X) = (x - X)^2.$$
- the median is elicitable with
$$s(x, X) = |x - X|.$$
- VaR_p is elicitable with
$$s(x, X) = (1 - p)(x - X)_+ + p(X - x)_+$$
if X has continuous inverse cdf at p.
- e_p is elicitable with
$$s(x, X) = (1 - p)(x - X)_+^2 + p(X - x)_+^2.$$

Elicitability and comparison

- Suppose observations are iid
- The estimated/modeled value of ρ is ρ_0 at $t = 0$;
- based on new iid observations X_t, $t > 0$, consider the statistics $s(\rho_0, X_t)$; for instance, test statistic can typically be chosen as $T_n(\rho_0) = \frac{1}{n}\sum_{t=1}^{n} s(\rho_0, X_t)$;
- $T_n(\rho_0)$: a statistic which indicates the goodness of forecasts.
- updating ρ: look at a minimizer for $T_n(\rho)$;
- the above procedure is model-independent.

Estimation procedures of an elicitable risk measure are straightforward to compare.

Perspective of a Regulator

Elicitability and regulation

- A value of risk measure ρ_0 is reported by a financial institution based on internal models.
- A regulator does not have access to the internal model, and she does not know whether ρ_0 is calculated honestly.
- She applies $s(\rho_0, X_t)$ as a daily penalty function for the financial institution. She may also compare it with a standard model chosen by the regulator.
- If the institution likes to minimize this penalty, it has to report the true value of ρ and use the most realistic model.
- the above procedure is model-independent.

VaR vs ES: elicitability

Theorem: Gneiting, 2011, JASA

Under some regularity conditions,

- VaR is elicitable;
- ES is not elicitable.

Backtestability

The unpublished idea was presented by Carlo Acerbi (MSCI). It is slightly modified.

Definition

A risk measure ρ is backtestable if there exists a function $Z : \mathbb{R}^2 \to \mathbb{R}$ such that for each $X \in \mathcal{X}$,

$$x \mapsto \mathbb{E}[Z(x, X)] \text{ is increasing, and}$$

$$\mathbb{E}[Z(x, X)] < 0 \text{ for } x < \rho(X) \text{ and } \mathbb{E}[Z(x, X)] > 0 \text{ for } x > \rho(X).$$

That is, zero can be used as a benchmark to distinguish whether a risk measure is underestimated. This is because a regulator is mainly concerned about underestimation.

Again we assume all integrals are finite in the following.

Proposition (*)

Suppose that a standardized risk measure ρ is backtestable, then it is elicitable with a score function convex in its first argument.

- One can always choose

$$s(x,y) = \int_y^x Z(t,y)\mathrm{d}t.$$

  Equivalently, $\partial s(x,y)/\partial x = Z(x,y)$.

- Assuming X is has continuous cdf at p, VaR_p is backtestable with

$$Z(x,y) = -\mathrm{I}_{\{x<y\}}p + \mathrm{I}_{\{x>y\}}(1-p).$$

Elicitability

Remarks: the relevance of elicitability for risk management purposes is heavily contested:

- McNeil, Frey and Embrechts (2005): backtesting of ES is possible (semi-parametric EVT models)
- Emmer, Kratz and Tasche (2014): alternative method for backtesting ES
- Davis (2016): backtesting based on prequential principle

Shortfall Risk Measures

Recall the definition of shortfall risk measures:

$$\rho(X) = \inf\{x \in \mathbb{R} : \mathbb{E}[\ell(X - x)] \le \ell_0\}.$$

ℓ: an increasing function, called a loss function. ρ is a convex risk measure if and only if ℓ is convex. We assume ℓ to be strictly increasing.

Proposition (*)

A shortfall risk measure is always elicitable and backtestable.

- Take $Z(x, y) = \ell_0 - \ell(y - x)$.

Convex level set

An interesting related property for law-determined risk measures is having convex level sets. Let F_X be the distribution function of $X \in \mathcal{X}$.

[CL] Convex level sets: If $\rho(X) = \rho(Y)$, then $\rho(Z) = \rho(X) = \rho(Y)$ for all $\lambda \in [0,1]$ and $F_Z = \lambda F_X + (1-\lambda)F_Y$.

Proposition (*)

An elicitable risk measure always has convex level sets.

Corollary

A shortfall risk measure always has convex level sets.

Eventually, it was established that among convex risk measures, [CL] characterizes convex shortfall risk measures.

Theorem (Delbaen-Bellini-Bignozzi-Ziegel 2016 FS)

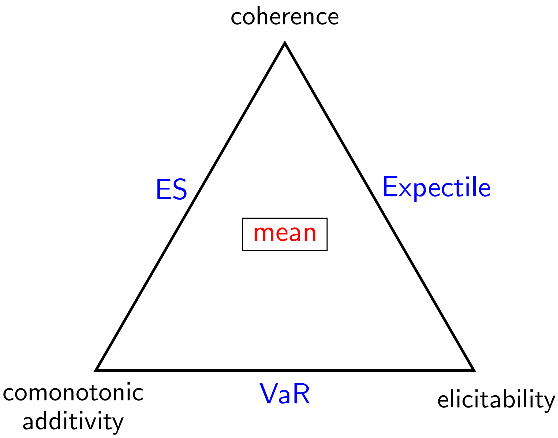
A law-determined convex risk measure on L^∞ satisfies [CL] if and only if it is a convex shortfall risk measure.

Elicitable Risk Measures

Some results

- if ρ is coherent, comonotonic additive and elicitable, then ρ is the mean (Ziegel, 2015);
- if ρ is comonotonic additive and elicitable, then ρ is a VaR or the mean (Kou and Peng, 2014; Wang and Ziegel, 2015);
- if ρ is coherent and elicitable, then ρ is an expectile (Delbaen, Bellini, Bignozzi and Ziegel, 2016);
- if ρ is convex and elicitable, then ρ is a convex shortfall risk measure (Delbaen, Bignozzi, Bellini and Ziegel, 2016).

Triangle of Risk Measures



Change of Currency

- There are two currencies (domestic and foreign).
- The exchange rate at future time T from the domestic currency to the foreign currency is denoted by R_T.
- In practice, R_T is random.
- Suppose that the random loss/profit at time T of a financial institution is X (in domestic currency).

Change of Currency

Let ρ be a monetary risk measure.

- A regulator uses an acceptance set $\mathcal{A}_\rho$ to determine the solvency of this financial institution.
  - The institution is solvent if $X \in \mathcal{A}_\rho$.
- Another regulator uses the same acceptance set $\mathcal{A}_\rho$, but it is calculated based on the foreign currency.
  - The institution is solvent if $\frac{R_T}{R_0} X \in \mathcal{A}_\rho$.
- Both solvency criteria should be equivalent; that is, for $R = R_T/R_0$, one should have $X \in \mathcal{A}_\rho \Rightarrow RX \in \mathcal{A}_\rho$.

Change of Currency

For a risk measure ρ:

[EI] Exchange-invariance: for $X \in \mathcal{X}$, if $\rho(X) \leq 0$, then $\rho(RX) \leq 0$ for all positive random variables $R \in \mathcal{X}$.

Proposition (*)

If a monetary risk measure satisfies [EI], then it satisfies [PH].

- [EI] is a very strong property.

Change of Currency

Some simple results:

Theorem (Koch-Medina-Munari 2016 JBF*)

For $p \in (0,1)$, VaR_p satisfies [EI] and ES_p does not satisfy [EI].

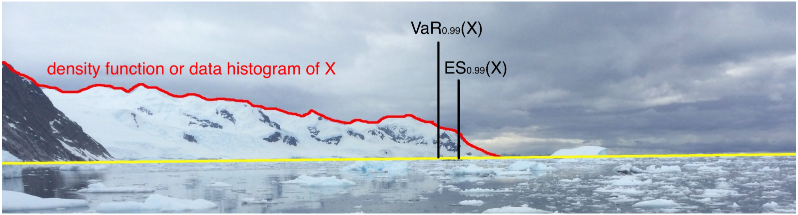
- ES has currency issues as a global regulatory risk measure.

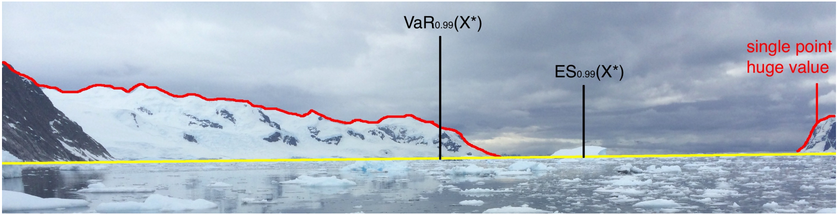
Robust Statistics

Robustness addresses the question of "what if the data is compromised with small error?" (e.g. outlier)

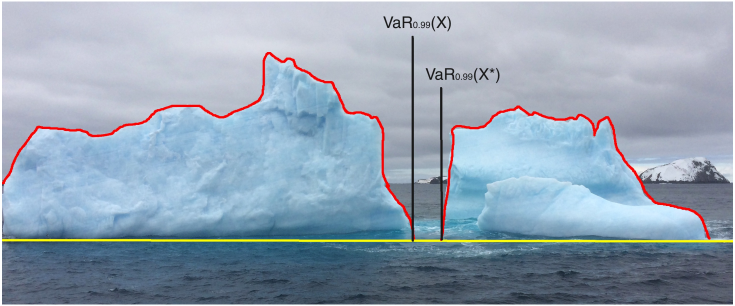
- Originally robustness was defined on estimators (of a quantity T)
- Would the estimation be ruined if an outlier is added to the sample?
  - Think about VaR and ES non-parametric estimates

VaR and ES Robustness





VaR and ES Robustness



- Non-robustness of VaR_p only happens if the quantile has a gap at p
- Is this situation relevant for risk management practice?

Robust Statistics

Classic qualitative robustness:

- Hampel (1971 AoMS): the robustness of an estimator of T is equivalent to the continuity of T with respect to underlying distributions (both with respect to the same metric)

When we talk about the robustness of a statistical functional, (Huber-Hampel's) robustness typically refers to continuity with respect to some metric.

General reference: Huber and Ronchetti, 2007 book

Consider the continuity of $\rho : \mathcal{X} \to \mathbb{R}$.

- The strongest sense of continuity is w.r.t. weak convergence.
  - $X_n \to X$ weakly, then $\rho(X_n) \to \rho(X)$.
- Quite restrictive
- Practitioners like weak convergence

In Part II, we have seen a few different types of continuity for risk measures.

With respect to weak convergence:

- VaR_p is continuous at distributions whose quantile is continuous at p. VaR_p is argued as being almost robust.
- ES_p is not continuous for any $\mathcal{X} \supset L^\infty$

ES_p is continuous w.r.t. some other (stronger) metric, e.g. L^q, $q \geq 1$ metric (or the Wasserstein-L^p metric)

Take $\mathcal{X} = L^\infty$. From weak to strong:

- Continuity w.r.t. L^∞ convergence: all monetary risk measures
- Continuity w.r.t. L^q, $q \geq 1$ convergence: e.g. ES_p, $p \in (0, 1)$
- Continuity w.r.t. weak convergence (a.s. or in probability): (almost) VaR_p, $p \in (0, 1)$. A convex risk measure cannot be continuous with respect to a.s., $\mathbb{P}$ or weak convergence.

For distortion risk measures:

- A distortion risk measure is continuous on L^∞ iff its distortion function h has a (left and right) derivative which vanishes at neighbourhoods of 0 and 1 (classic property of L-statistics; see Cont-Deguest-Scandolo 2010 QF).

Some references and related papers:

- Bäuerle-Müller (2006 IME)
- Stahl-Zheng-Kiesel-Rühlicke (2012 SSRN)
- Krätschmer-Schied-Zähle (2012 JMVA, 2014 FS, 2015 arXiv)
- Embrechts-Wang-Wang (2015 FS)
- Cambou-Filipović (2016+ MF)
- Daníelsson-Zhou (2015 SSRN)

Example

Example: different internal models

- Same data set, two different parametric models (e.g. normal vs student-t).
- Estimation of parameters, and compare the VaR and ES for two models.
- VaR is more robust in this setting, since it does not take the tail behavior into account (normal and student-t do not make a big difference).
- ES is less robust (heavy reliance on the model's tail behavior).
- Capital requirements: heavily depends on the internal models.

End of Lecture Note

- The field of risk measures is developing really fast in both academia and industry.
- No grand conclusion can be made at this moment.
- Different situations require different principles, and judgement should always be made with caution.
- Uncertainty always exists.

Thank you for attending the lectures!