Short Course

Theory and Practice of Risk Measurement

Part 3

Law-determined Risk Measures

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Part 3

- Law-determined risk measures
- Shortfall risk measures
- Comonotonicity
- Distortion risk measures
- Risk measures and risk-aversion

Law-determined Risk Measures

In this part of the lectures, we study an important subclass of "simplified" risk measures. This class of risk measures is determined by the distribution of a random loss.

- Such risk measures are referred to as law-determined risk measures.
- All previous examples are in fact law-determined risk measures.

Formally, the property[1] is

[LD] law-determination: $\rho(X) = \rho(Y)$ if $X, Y \in \mathcal{X}$, $X \stackrel{\mathrm{d}}{=} Y$.

Here, we emphasize that the reference (real-world) probability measure $\mathbb{P}$ is important in [LD], since the distributions of X and Y depend on $\mathbb{P}$.

- In Mathematical Finance this property is often called "law-invariance".

[1] We use $X \stackrel{\mathrm{d}}{=} Y$ to indicate that the distribution functions $\mathbb{P}(X \leq \cdot)$ and $\mathbb{P}(Y \leq \cdot)$ are identical.

Remark: In all previous properties, namely [M], [CI], [SA], [PH], [CX], and [FP], the reference probability measure $\mathbb{P}$ is irrelevant. If we state them under another measure Q which is equivalent to $\mathbb{P}$, the properties will not change.

Very often from a statistical consideration, we may only know about the distribution of a risk, but not the mapping $X : \Omega \to \mathbb{R}$.

- law-determined functionals are thereby also often called statistical functionals
- In many practical situations one learns about a risk from simulation instead of probabilistic characteristics
- it significantly reduces the cardinality of the set of risk measures at study:
  - The set of risks $\mathcal{X}$: $\mathcal{F}$-measurable functions: $\Omega \to \mathbb{R}$
  - The set of distributions $\mathcal{D}$: increasing functions: $\mathbb{R} \to [0, 1]$
- a risk may not be well-described by its law: for instance, lottery vs insurance

Law-determined Risk Measures

- In this part of the lecture, we continue to take $\mathcal{X} = L^\infty$ as the standard set of risks to consider.
- It turns out that the class of distortion risk measures, including VaR and ES, is a crucial part in the study of law-determined risk measure.
- VaR and ES are particularly important and they have unique roles to play.

Shortfall Risk Measures

Shortfall risk measures:

$$\rho(X) = \inf\{y \in \mathbb{R} : \mathbb{E}[\ell(X - y)] \leq \ell_0\}.$$

ℓ: an increasing function, called a loss function. ℓ is typically convex. $\ell_0 \in \mathbb{R}$ and usually can be taken as $\ell(0)$.

- It is easy to verify that ρ is a monetary risk measure.
- Motivated from indifference pricing: the acceptance set of ρ is

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \mathbb{E}[\ell(X)] \leq \ell_0\}.$$

- If ℓ is convex, then ρ is a convex risk measure.

Exercise: verify that VaR_p, $p \in (0,1)$ is a shortfall risk measure with loss function

$$\ell(x) = \mathrm{I}_{\{x>0\}} - (1-p), \quad \ell_0 = 0.$$

Entropic Risk Measures

Take $\ell(x) = e^{\beta x}$, $\beta > 0$ and $\ell_0 = 1$ in a shortfall risk measure, then ρ becomes the entropic risk measure:

$$\mathrm{ER}_\beta(X) = \frac{1}{\beta} \log \mathbb{E}[e^{\beta X}].$$

(also known as exponential principle in actuarial science)

- ER_β is neither subadditive or positively homogeneous
- It is the only convex (non-coherent) risk measure which has an "explicit" form
- It can also be generated by an exponential utility (cf. Assignment 1)
- $\mathrm{ER}_\beta(X) \geq \mathbb{E}[X]$ from Jensen's inequality
- ER_β satisfies independent additivity: if X and Y are independent, then $\mathrm{ER}_\beta(X+Y) = \mathrm{ER}_\beta(X) + \mathrm{ER}_\beta(Y)$

Entropic Risk Measures

Entropic risk measure is a convex risk measure

- We can calculate its penalty function:

$$\alpha^{\min}(Q) = \sup_{X\in\mathcal{X}} \{\mathbb{E}^Q[X] - \frac{1}{\beta}\log \mathbb{E}^{\mathbb{P}}[e^{\beta X}]\}$$
$$= \frac{1}{\beta} \sup_{X\in\mathcal{X}} \{\mathbb{E}^Q[X] - \log \mathbb{E}^{\mathbb{P}}[e^{X}]\}$$
$$= \frac{1}{\beta}\mathbb{E}^Q\left[\log\left(\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\right)\right] = \frac{1}{\beta}H(Q|\mathbb{P}),$$

where $H(Q|\mathbb{P})$ is the relative entropy of Q with respect to $\mathbb{P}$.

  - The supremum is attained by $X = \log\left(\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\right)$; see Lemma 3.29 of Föllmer and Schied (2011).

Entropic Risk Measures

Entropic risk measure has a dual representation

$$\mathrm{ER}_\beta(X) = \sup_{Q \in \mathcal{P}} \left\{ \mathbb{E}^Q[X] - \frac{1}{\beta} H(Q|\mathbb{P}) \right\}.$$

- ρ penalizes on the relative entropy of Q, which is a natural consideration.
- By taking $Q = \mathbb{P}$, we can see that $\mathrm{ER}_\beta(X) \geq \mathbb{E}[X]$.

A coherent risk measure based on the relative entropy is the coherent entropic risk measure:

$$\mathrm{CER}_c(X) = \sup_{Q \in \mathcal{R}_c} \left\{ \mathbb{E}^Q[X] \right\}, \quad X \in \mathcal{X}$$

where $c > 0$ and $\mathcal{R}_c = \{Q \in \mathcal{P} : H(Q|\mathbb{P}) \leq c\}$.

Entropic Risk Measures

More on CER:

- Only scenarios that are not far away from $\mathbb{P}$ are considered - from this point of view, it is similar to an Expected Shortfall - one has $\mathrm{CER}_{c_p}(X) \geq \mathrm{ES}_p(X)$ where $c_p = -\log(1-p)$. Details: Föllmer and Knispel (2011).
- Connection:

$$\mathrm{CER}_c(X) = \min_{\beta > 0} \left\{ \mathrm{ER}_\beta(X) + \frac{c}{\beta} \right\}$$

- Shortcoming: computationally/statistically not straightforward

Expectiles

Take $\ell(x) = px_+ - (1-p)x_-$, $p \in (0,1)$ and $\ell_0 = 0$, then ρ becomes the p-expectile (well-defined on L^1):

$$e_p(X) = \min\{t \in \mathbb{R} : p\mathbb{E}[(X-t)_+] \leq (1-p)\mathbb{E}[(X-t)_-]\}.$$

An alternative formulation is (the following argmin is unique for $X \in L^2$):

$$e_p(X) = \arg\min_{t \in \mathbb{R}}\{p\mathbb{E}[(X-t)_+^2] + (1-p)\mathbb{E}[(X-t)_-^2]\}.$$

Expectiles

Let $\Omega(X) = \frac{\mathbb{E}[X_+]}{\mathbb{E}[X_-]}$ be the Omega ratio of X, then

$$e_p(X) = \min\left\{t \in \mathbb{R} : \Omega(X - t) \leq \frac{1-p}{p}\right\}.$$

The acceptance set of e_p is of this form

$$\mathcal{A}_{e_p} = \left\{X \in \mathcal{X} : \Omega(X) \leq \frac{1-p}{p}\right\}.$$

Expectiles

e_p is a coherent risk measure if and only if $p \geq 1/2$:

- originally proposed in the statistical literature: Newey and Powell (1987, Econometrika)
- expectiles are the only coherent risk measures which are elicitable: Ziegel (2015, MF), Delbaen et al. (2015, FS)
- representation of expectile: Bellini et al. (2014, IME).
- estimation is straightforward
- problem: interpretation is not easy and computation can be involved

Limit Behavior

Some limit behavior of risk measures: we interpret the parameter values at 0, 1, ∞ in the sense of limit.

1. $\mathrm{VaR}_0(X) = \text{ess-inf}(X)$, $\mathrm{VaR}_1(X) = \text{ess-sup}(X)$.
2. $\mathrm{ES}_0(X) = \mathbb{E}[X]$, $\mathrm{ES}_1(X) = \text{ess-sup}(X)$.
3. $\mathrm{ER}_0(X) = \mathbb{E}[X]$, $\mathrm{ER}_\infty(X) = \text{ess-sup}(X)$.
4. $\mathrm{CER}_0(X) = \mathbb{E}[X]$, $\mathrm{CER}_\infty(X) = \text{ess-sup}(X)$.
5. $e_0(X) = \text{ess-inf}(X)$, $e_{1/2}(X) = \mathbb{E}[X]$, $e_1(X) = \text{ess-sup}(X)$.

- At this moment, the natural question is to ask: what is a representation theorem for law-determined coherent risk measure ρ? It must have the form

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X]$$

for some appropriately chosen set $\mathcal{R}$ of probability measures. Note that not all choices of $\mathcal{R}$ would make ρ law-determined.

- Before we answer this question, we first look at some other interesting and relevant mathematical properties here: comotonicity and comonotonic additivity.

Comonotonicity

Definition

A pair of random variables $(X, Y) \in (L^0)^2$ is said to be comonotonic if there exists a random variable Z and two increasing functions f, g such that almost surely $X = f(Z)$ and $Y = g(Z)$.

- We also say "X and Y are comonotonic" when there is no confusion
- X and Y move in the same direction. This is a strongest (and simplest) notion of positive dependence.
- Two risks are not a hedge to each other if they are comonotonic
- We use $X /\!/ Y$ to represent that (X, Y) is comonotonic.

Comonotonicity

Some examples of comonotonic random variables:

- a constant and any random variable
- X and X
- X and $\mathrm{I}_{\{X \geq 0\}}$
- In the Black-Scholes framework, the time-t prices of a stock S and a call option on S

Note: in the definition of comonotonicity, the choice of $\mathbb{P}$ is irrelevant.

Theorem

The following are equivalent:

(i) *X and Y are comonotonic;*

(ii) *For some strictly increasing functions f, g, $f(X)$ and $g(Y)$ are comonotonic.*

(iii) $\mathbb{P}(X \leq x, Y \leq y) = \min\{\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)\}$ *for all* $(x, y) \in \mathbb{R}^2$;

(iv) $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ *for a.s.* $(\omega, \omega') \in \Omega \times \Omega$.

- If one is familiar with the theory of copulas, then two continuous random variables X and Y are comonotonic if and only if the copula of (X, Y) is the Fréchet upper copula.

In the following, the four random variables $X, Y, Z, W \in L^2$ satisfy $Z \overset{\mathrm{d}}{=} X$ and $W \overset{\mathrm{d}}{=} Y$.

Proposition

Suppose that X and Y are comonotonic. The following hold:

(i) $\mathbb{P}(X \le x, Y \le y) \ge \mathbb{P}(Z \le x, W \le y)$ *for all* $(x, y) \in \mathbb{R}^2$;

(ii) $\mathbb{E}[XY] \ge \mathbb{E}[ZW]$;

(iii) $\mathrm{Corr}(X, Y) \ge \mathrm{Corr}(Z, W)$.

Comonotonic Additivity

One more important property from an economic viewpoint:

[CA] comonotonic additivity: $\rho(X+Y)=\rho(X)+\rho(Y)$ if X and Y are comonotonic.

To interpret this property:

- If $X \,/\!\!/\, Y$, then they are not a hedge of each other. Therefore, one should not receive any diversification benefit from putting such risks together. This leads to $\rho(X+Y) \geq \rho(X)+\rho(Y)$.
- If one in addition asks for subadditivity, then we must have $\rho(X+Y)=\rho(X)+\rho(Y)$.

Comonotonic Additivity

[CA] is known in economic decision theory as the dual independence axiom (Yaari, 1987).

- Suppose that an investor has a preference (total order) among all risks in $\mathcal{X}$.
- Assume: if she prefers X over Y, then she should also prefer $X + Z_X$ over $Y + Z_Y$, where $Z_X \overset{\mathrm{d}}{=} Z_Y$, $Z_X /\!/ X$, $Z_Y /\!/ Y$. That is, adding identically distributed comonotonic partners to two risks respectively does not change the preference between them.

Comonotonic Additivity

If this investor use a law-determined risk measure ρ to rank risks, we can write the dual independence axiom as a formal property of ρ:

[DI] If $\rho(X) \le \rho(Y)$, then $\rho(X + Z_X) \le \rho(Y + Z_Y)$ for $Z_X \overset{\mathrm{d}}{=} Z_Y$, $Z_X \,/\!\!/\, X$, $Z_Y \,/\!\!/\, Y$.

Proposition (*)

For a law-determined monetary risk measure ρ with $\rho(0) = 0$, [CA] and [DI] are equivalent.

Comonotonic Additivity

Historical remark:

- In Mathematical Finance, Analytical study of comonotonic additive risk measures started around 2000: Kusuoka (2001).
- In Actuarial Science, insurance premium principles with comonotonic additivity was studied since 1995: Wang, Young and Panjer (1997).
- In Economic Decision Theory, the dual independence axiom and its equivalent forms have been studied since the 1980s: Schmeidler (1986, 1989), Yaari (1987), Denneberg (1990, 1994 book).

Comonotonic Additivity

In general, [CA] is a very strong property.

Proposition (*)

For a monetary risk measure ρ, [CA] implies [PH].

Examples:

- VaR_p, $p \in (0,1)$ is comonotonic additive.
- ES_p, $p \in (0,1)$ is comonotonic additive.
- ER_β, $\beta \in (0,\infty)$ is not comonotonic additive.
- e_p, $p \in (0,1) \setminus \{1/2\}$ is not comonotonic additive. (Hence, the converse of the proposition is not true.)

Exercise. For $p \geq 1/2$, take a $\mathrm{U}[0,1]$ random variable U, $X = \mathrm{I}_{\{U \geq 1/2\}}$ and $Y = \mathrm{I}_{\{U \geq p\}}$.

- Observe that X and Y are comonotonic.
- Calculate the value of $e_p(X)$, $e_p(Y)$ and $e_p(X+Y)$.
- Show that $e_p(X+Y) = e_p(X) + e_p(Y)$ if and only if $p = 1/2$.

Hence, e_p, $p > 1/2$ is not comonotonic additive.

- From this exercise we may also notice how unfriendly the calculation of $e_p(X)$ is even for some simplest distributions.

Distortion Risk Measures

Theorem: Yaari, 1987; Wang, Young and Panjer, 1997

A law-determined and comonotonic additive monetary risk measure ρ has the following representation:

$$\rho(X) = \rho_h(X) := \int_{\mathbb{R}} x \mathrm{d}h(F(x)), \ X \in \mathcal{X}, \ X \sim F$$

where h is an increasing function on $[0,1]$ with $h(0) = 0$ and $h(1) = 1$.

ρ_h is called a distortion risk measure (DRM). h is the distortion function of ρ_h.

- ES and VaR are special cases of distortion risk measures.
- The proof is a standard property of Choquet integrals.

Distortion Risk Measures

Alternative representation

If $X \sim F$ and F^{-1} is a continuous function on $[0,1]$, then a distortion risk measure ρ_h can be written as

$$\rho_h(X) = \int_0^1 \mathrm{VaR}_p(X) \mathrm{d}h(p),$$

where h is a distribution function on $[0,1]$.

- VaR_p: h is a point mass at p
- ES_p: h is the uniform distribution over $[p,1]$
- We will work with this representation for simplicity

Distortion Risk Measures

Distortion risk measures are very popular due to its advantages in

- comonotonic additivity
- economic interpretations
- estimation
- computation

Distortion Risk Measures

For a continuous and strictly increasing utility function u and a distortion function h, the rank-dependent expected utility (RDEU) is defined as

$$U_{u,h}(X) = \rho_h(u(X)).$$

- RDEU theory is one of the most popular generalizations of the expected utility theory, and could explain the Allais paradox.
- See Quiggin (1982, 1993).

Distortion Risk Measures

Proposition

(a) *For a distortion risk measure* ρ_h *and* $X \in L^\infty$, $X \sim F$ *where* F^{-1} *is continuous,*

$$\rho_h(X_F) = F^{-1}(0) + \int_0^1 (1 - h(t))\mathrm{d}F^{-1}(t).$$

(b) *For two distortion risk measures* ρ_{h_1}, ρ_{h_2},

$$h_1(t) \le h_2(t) \ \ \forall t \in [0,1] \quad \Leftrightarrow \quad \rho_{h_1}(X) \ge \rho_{h_2}(X) \ \ \forall X \in L^\infty.$$

Coherent Distortion Risk Measures

- A distortion risk measure is always positively homogeneous.
- It is not necessarily convex or subadditive (VaR for instance).
- It needs convexity or subadditivity to be a coherent risk measure.
- The next question is: what distortion function h would give a subadditive (coherent) distortion risk measure.

Theorem: Kuosuka, 2001

A law-invariant and comonotonic additive coherent risk measure ρ (with the Fatou property) has the following two representations:

(i) $$\rho(X) = \rho_g(X) = \int_0^1 \mathrm{VaR}_p(X)\mathrm{d}g(p),\ X \in \mathcal{X},$$
where g is a convex distribution function on $[0,1]$;

(ii) $$\rho(X) = \rho_h^*(X) := \int_0^1 \mathrm{ES}_p(X)\mathrm{d}h(p),\ X \in \mathcal{X},$$
where h is a distribution function on $[0,1]$.

To connect them, one has

$$g'(q) = \int_0^q \frac{1}{1-p}\mathrm{d}h(p).$$

- In different context, under different conditions, with different focuses, the previous theorem and some similar forms can be found in Schmeidler (1986); Denneberg (1994); Wang (1996); Kusuoka (2001); Acerbi (2002).

Law-determined Coherent Risk Measures

Theorem: Kusuoka, 2001*

A law-determined coherent risk measure (with the Fatou property) has the following representation:

$$\rho(X) = \sup_{h \in \mathcal{R}_I} \int_0^1 \mathrm{ES}_p(X) \mathrm{d}h(p), \; X \in \mathcal{X}$$

where $\mathcal{R}_I$ is a collection of probability measures on $[0, 1]$.

- That is, ρ is the supremum of a class of coherent distortion risk measures.
- This result is called Kusuoka's representation.

Law-determined Convex Risk Measures

Theorem: Frittelli-Rosazza Gianin, 2005, AME

A law-determined convex risk measure (with the Fatou property) has the following representation

$$\rho(X) = \sup_{h \in \mathcal{P}_I} \left\{ \int \mathrm{ES}_p(X) \mathrm{d}h(p) - \alpha(h) \right\}, \quad X \in \mathcal{X}$$

where $\mathcal{P}_I$ is the set of probability measures on $[0,1]$, and $\alpha : \mathcal{P}_I \to (-\infty, \infty]$ is a penalty function.

- This is very similar to Kusuoka's representation of law-determined coherent risk measures.

Fatou Property

Theorem: Jouini-Schachermayer-Touzi, 2006, AME

A law-determined convex risk measure on L^∞ has the Fatou property.

- Same conclusion holds true for risk measures on L^q, $q \in [1, \infty)$ which takes values in $\mathbb{R}$, although we have not formally defined the Fatou's property on L^q
- Extension to L^q: Kaina and Rüschendorf (2009, MMOR) and Filipović and Svindland (2012, MF).
- We may remove "(with the Fatou property)" in the previous few theorems.

In the next, we consider the relationship between risk measures and the notion of risk-aversion.

Definition (Convex order)

For $X, Y \in L^1$, X is smaller than Y in (resp. increasing) convex order, denoted as $X \prec_{\mathrm{cx}} Y$ (resp. $X \prec_{\mathrm{icx}} Y$), if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all (resp. increasing) convex functions f such that the expectations exist.

Convex Order

- Increasing convex order describes a preference among risks for risk-averse investors (called second-order stochastic dominance (SSD) in decision theory)
- a risk-averse investor prefers a risk with less variability (uncertainty) against one with larger variability, and she prefers a risk with a certainly smaller loss against a risk with a larger loss
- convex order and increasing convex order are based on the law of random variables

Convex Order

Some examples and properties (all random variables are in L^1):

- $X \prec_{\mathrm{cx}} Y$ implies $X \prec_{\mathrm{icx}} Y$.
- $X \le Y$ a.s. implies $X \prec_{\mathrm{icx}} Y$.
- $\mathbb{E}[X|\mathcal{G}] \prec_{\mathrm{cx}} X$ for any σ-field $\mathcal{G}$. In particular, $\mathbb{E}[X] \prec_{\mathrm{cx}} X$.
- If $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $X = aY$, $a > 1$, then $Y \prec_{\mathrm{cx}} X$.
- If $X \prec_{\mathrm{icx}} Y$, $Y \prec_{\mathrm{icx}} Z$, then $X \prec_{\mathrm{icx}} Z$.
- If $X \prec_{\mathrm{icx}} Y$, then $f(X) \prec_{\mathrm{icx}} f(Y)$ for any increasing function f.
- If $X \overset{\mathrm{d}}{=} Z$, $Y \overset{\mathrm{d}}{=} W$, $X /\!\!/ Y$, then $Z + W \prec_{\mathrm{cx}} X + Y$.
- $X \prec_{\mathrm{cx}} Y$ if and only if $\mathrm{ES}_p(X) \le \mathrm{ES}_p(Y)$ for all $p \in (0, 1)$.

We will focus on the following property:

[SC] SSD consistentcy: $\rho(X) \le \rho(Y)$ if $X \le_{\mathrm{icx}} Y$, $X, Y \in \mathcal{X}$.

Similar properties:

[CC] Convex order consistency: $\rho(X) \le \rho(Y)$ if $X \prec_{\mathrm{cx}} Y$, $X, Y \in \mathcal{X}$.

[DM] Dilatation monotonicity: $\rho(X) \le \rho(Y)$ if $(X, Y) \in \mathcal{X}^2$ is a martingale.

[DC] Diversification consistency: $\rho(X + Y) \le \rho(X^c + Y^c)$ if $X, Y, X^c, Y^c \in \mathcal{X}$, $X \overset{\mathrm{d}}{=} X^c$, $Y \overset{\mathrm{d}}{=} Y^c$, and $X^c /\!/ Y^c$.

Risk Aversion in Risk Measures

[SC] is sometimes called strong risk aversion in economic decision theory.

- [SC] is consistent with an aversion to mean-preserving spreads, and it implies that $\rho(\mathbb{E}[X]) \leq \rho(X)$ for all $X \in \mathcal{X}$.
- It is natural to require [SC] for a risk measure implemented in regulation, as this will encourage financial institutions to make decisions consistent with the common notion of risk aversion.
- [DC] implies that the undiversified portfolio has a larger capital requirement. This represents the third notion of diversification benefit in addition to subadditivity and convexity (cf. Part II).

In fact [CC], [SC], [DM] and [DC] are equivalent for law-determined risk measures on L^∞.

Proposition (Mao-Wang, 2016)

For a risk measure ρ on L^∞, the following are equivalent:

(a) *ρ satisfies [CI] and [SC];*

(b) *ρ satisfies [M], [CI] and [CC];*

(c) *ρ satisfies [M], [CI] and [DM];*

(d) *ρ satisfies [M], [CI] and [DC].*

Moreover, each case implies that ρ satisfies [LD].

- The implication [DM]$\Rightarrow$[LD] is very tricky. See Cherny and Grigoriev (2007, FS).

Since [SC] is consistent with the use of concave utility function, one may expect that convex risk measures have the same property. This is indeed true.

Theorem (*)

A law-determined convex risk measure on L^∞ with the Fatou property satisfies [SC].

- This result (when $\mathcal{X} = L^\infty$) is first given in the 2004 version of Föllmer and Schied (2011). Generalizations to L^q can be found in Svindland (2008).

Recently we are able to characterize all risk measures that satisfies [SC].

Theorem: Mao-Wang, 2016

A monetary risk measure ρ on L^∞ satisfies [SC] if and only if it has the following representation:

$$\rho(X) = \inf_{\tau \in \mathcal{C}} \tau(X)$$

where $\mathcal{C}$ is a collection of law-determined convex risk measures.

Finally, the following representation is available.

Theorem: Mao-Wang 2016

A monetary risk measure ρ on L^∞ satisfies [SC] if and only if it has the following representation

$$\rho(X) = \inf_{g \in G} \sup_{\alpha \in [0,1]} \left\{ \mathrm{ES}_\alpha(X) - g(\alpha) \right\}, \quad X \in L^\infty, \tag{1}$$

where G is a set of functions mapping $[0,1]$ to $\mathbb{R}$. Moreover, a risk measure ρ is a law-determined convex (coherent) risk measure if and only if it has a representation (1) in which G is a convex set (cone).

Kusuoka representations

A risk measure ρ on L^∞:

$$\text{[TI]+[SC]} = \inf_{\alpha\in\mathcal{V}} \sup_{h\in\mathcal{P}_I} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) - \alpha(h) \right\} \qquad \text{for some set } \mathcal{V}$$

$$\xrightarrow{+\text{[CX]}} \sup_{h\in\mathcal{P}_I} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) - \alpha(\mu) \right\} \qquad \text{for some function } \alpha$$

$$\xrightarrow{+\text{[PH]}} \sup_{h\in\mathcal{R}_I} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) \right\} \qquad \text{for some set } \mathcal{R}_I \subset \mathcal{P}_I$$

$$\xrightarrow{+\text{[CA]}} \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) \qquad \text{for some } h \in \mathcal{P}_I.$$

Remark: [TI]+[SC]+[CA] is sufficient for the last representation