Risk Aversion, Insurance Propensity, and Risk Measures

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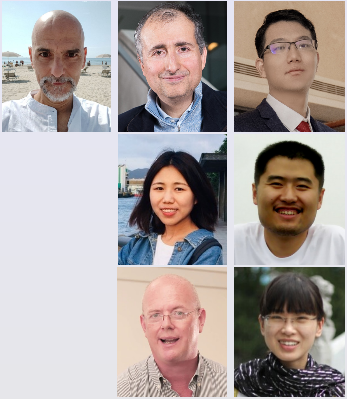
Content



- Maccheroni/Marinacci/W./Wu
  Risk aversion and insurance propensity
  arXiv: 2310.09173, 2023
- Han/Wang/W./Wu
  Risk concentration and the mean-Expected Shortfall criterion
  Mathematical Finance, 2023
- Bellini/Mao/W./Wu
  Duet expectile preferences
  Working paper, 2023

Agenda

Risk aversion

Risk aversion

- Probability and gambling — e.g., Bernoulli 1738 ['54 ECMA]
- Finance — e.g., Markowitz'52 JF; Merton'73 ECMA
- Insurance — e.g., Arrow'63 AER
- Economics — e.g., Pratt'64 ECMA
- Psychology — e.g., Kahneman/Tversky'79 ECMA
- Experimental observations — e.g., Holt/Larry'02 AER

What is risk aversion?

Let $\succsim$ be a preference relation over random payoffs on (S, Σ, P)

- Arrow'63; Pratt'64 — Weak risk aversion

$$\mathbb{E}(f) \succsim f$$

- Rothschild/Stiglitz'70 — Strong risk aversion

$$f \geq_{\mathrm{cv}} g \implies f \succsim g$$

- $f \geq_{\mathrm{cv}} g$ means $\mathbb{E}(\varphi \circ f) \geq \mathbb{E}(\varphi \circ g)$ for all concave φ

What is risk aversion?

- The expected utility (EU) theory von Neumann/Morgenstern'44

$$f \succsim g \iff \int u \circ f \mathrm{d}P \geq \int u \circ g \mathrm{d}P$$

for an increasing $u : \mathbb{R} \to \mathbb{R}$

- In the EU framework

concavity of u $\iff$ strong risk aversion $\iff$ weak risk aversion

What is risk aversion?

- The dual utility (DU) theory (Choquet integral) Yaari'87 ECMA

$$f \succsim g \iff \int f \mathrm{d}(\phi \circ P) \geq \int g \mathrm{d}(\phi \circ P)$$

for an increasing $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$

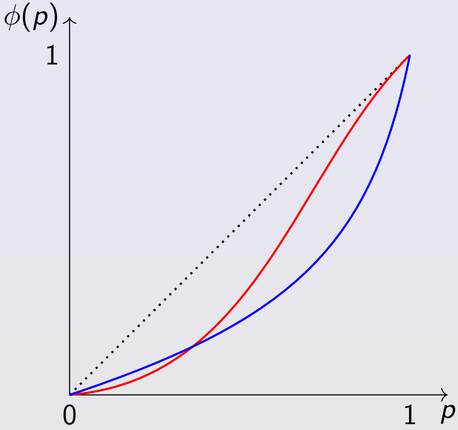
- In the DU framework

convexity of $\phi \iff$ strong risk aversion

$\phi \leq$ identity $\iff$ weak risk aversion

- Generally: strong $\implies$ weak; the converse is not true

What is risk aversion?



Insurance propensity

Insurance propensity

A merchant is about to ship commodities with a vessel

- The merchant earns $a > 0$ if the vessel reaches destination (state ω_1), otherwise (state ω_2) loses $b > 0$

- The uncertain wealth of the merchant is denoted by

$$w = (a, -b)$$

- Assume that ω_1 and ω_2 are known to be equally likely

Insurance propensity

- Let $c, d > 0$. An insurance against the shipping failure

$$f = (-c, d)$$

- Another act g with $g \stackrel{\mathrm{d}}{=} f$ is

$$g = (d, -c)$$

(a gamble on the shipping success)

- A choice seems natural:

$$w + \underbrace{f}_{\text{insurance}} \succsim w + g$$

- Can this say anything about the risk attitude?

Insurance propensity

Basic framework

- (S, Σ, P): probability space, nonatomic or uniform on finite S
- $\mathcal{F}$: all Σ-measurable bounded real-valued functions
  - All results work also on the set $\mathcal{M}^\infty$ of all measurable functions with all finite moments (e.g., normal)
- Two random payoffs f and g are equally distributed, written $f \overset{\mathrm{d}}{=} g$, if $P \circ f^{-1} = P \circ g^{-1}$
- A binary relation $\succsim$ on $\mathcal{F}$ is a risk preference when it is a preorder such that

$$f \overset{\mathrm{d}}{=} g \Longrightarrow f \sim g$$

Classic notions of risk attitude

A preference $\succsim$ is

(i) strongly risk averse if, for all $f, g \in \mathcal{F}$, $f \geq_{\mathrm{cv}} g \Longrightarrow f \succsim g$;

(ii) strongly risk propense if, for all $f, g \in \mathcal{F}$, $f \geq_{\mathrm{cv}} g \Longrightarrow g \succsim f$;

(iii) risk neutral if, for all $f \in \mathcal{F}$, $\mathbb{E}(f) \sim f$;

(iv) weakly risk averse if, for all $f \in \mathcal{F}$, $\mathbb{E}(f) \succsim f$;

(v) weakly risk propense if, for all $f \in \mathcal{F}$, $f \succsim \mathbb{E}(f)$.

Risk neutrality $\iff$ strong risk aversion + strong risk propension

(also holds for the weak versions)

Insurance propensity

Given any initial wealth w, a random payoff f is:

(i) a full insurance for w, written $f \in \mathcal{I}^{\mathrm{fi}}(w)$, when

$$f = -w - \pi$$

for some premium $\pi \in \mathbb{R}$;

(ii) a proportional insurance for w, written $f \in \mathcal{I}^{\mathrm{pr}}(w)$, when

$$f = -(1-\varepsilon)\,w - \pi$$

for some premium $\pi \in \mathbb{R}$ and percentage excess $\varepsilon \in [0,1)$;

(iii) a deductible-limit insurance for w, written $f \in \mathcal{I}^{\mathrm{dl}}(w)$, when

$$f = (-w-\delta)^{+} \wedge \lambda - \pi$$

for some premium $\pi \in \mathbb{R}$, deductible $\delta \in \mathbb{R}$ and limit $\lambda \geq 0$.

Insurance propensity

$$\mathcal{I}^{\mathrm{fi}}(w) = \mathcal{I}^{\mathrm{pr}}(w) \cap \mathcal{I}^{\mathrm{dl}}(w)$$

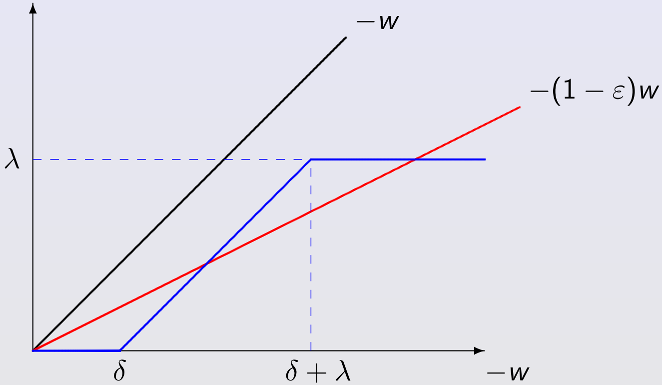


Figure: Proportional insurance (in red) and deductible-limit insurance (in blue) for loss $-w$

Insurance propensity

A risk preference $\succsim$ is:

(i) propense to full insurance when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\mathrm{fi}}(w) \implies w + f \succsim w + g;$$

(ii) propense to proportional insurance when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\mathrm{pr}}(w) \implies w + f \succsim w + g;$$

(iii) propense to deductible-limit insurance when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\mathrm{dl}}(w) \implies w + f \succsim w + g.$$

Risk-insurance equivalence

Risk-insurance equivalence

Propension to full insurance:

$$-\pi = w + f \succsim w + g \text{ where } g \overset{\mathrm{d}}{=} f = -w - \pi$$

Theorem 1

The following properties are equivalent for a risk preference:

(i) *weak risk aversion;*

(ii) *propension to full insurance.*

- (i)⇒(ii) is simple
- To show (ii)⇒(i), one needs to show $\mathbb{E}(f) \succsim f$ for all f from $-\pi \succsim w + g$ for all $g \overset{\mathrm{d}}{=} -w - \pi$
- For each f, need to find $g' \overset{\mathrm{d}}{=} g - \mathbb{E}(f)$ such that $f \overset{\mathrm{d}}{=} g - g'$

A key step to prove Theorem 1

Denote the essential supremum and the essential infimum of f by

$$u_f = \inf\{x \in \mathbb{R} : P(f \geq x) \geq 1\}, \ \ell_f = \inf\{x \in \mathbb{R} : P(f \geq x) > 0\}$$

Theorem 2

Let $k \geq 1$ and $f \in L^k$. Then $\mathbb{E}(f) = 0$ if and only if there exist $g, g' \in L^{k-1}$ such that $g \stackrel{\mathrm{d}}{=} g'$ and $g - g' \stackrel{\mathrm{d}}{=} f$. If, in addition, $f \in L^\infty$, then we can take $g, g' \in L^\infty$ satisfying $\ell_f \leq g, g' \leq u_f$.

Simple version: For $f \in L^\infty$,

$$\mathbb{E}(f) = 0 \iff f \stackrel{\mathrm{d}}{=} g - g' \text{ for some } g, g' \in L^\infty \text{ with } g \stackrel{\mathrm{d}}{=} g'$$

Some mathematics

Proof sketch. $\Leftarrow$:

- Use this

The Annals of Probability
1977, Vol. 5, No. 1, 157–158

AN UNEXPECTED EXPECTATION

BY GORDON SIMONS[1]

University of North Carolina

It is shown that, while the value of the expectation $E(X+Y)$ always depends on the random variables X and Y only through their marginal distributions, the same kind of statement cannot be made for $E(X+Y+Z)$.

- $\mathbb{E}(f) = \mathbb{E}(g - g') = \mathbb{E}(g - g^*)$ for $g^* \stackrel{\mathrm{d}}{=} g'$
- Take $g^* = g$

Some mathematics

Proof sketch (continued). $\Rightarrow$: for a finite uniform space:

- Let f have mean 0 and write $x_i = f(\omega_i)$

	ω_1	ω_2	$\cdots$	ω_{n-1}	ω_n
f	x_1	x_2	$\cdots$	x_{n-1}	x_n
g	x_1	$x_1 + x_2$	$\cdots$	$\sum_{i=1}^{n-1} x_i$	$\sum_{i=1}^{n} x_i$
g'	0	x_1	$\cdots$	$\sum_{i=1}^{n-2} x_i$	$\sum_{i=1}^{n-1} x_i$

- $\sum_{i=1}^{n} x_i = 0$
- The range statement can be shown by rearranging ω
- In the general case, g has one less moment than f

Expected utility

Example (EU).

- Suppose that $\succsim$ is EU with (measurable) utility function u
- Take $a \in \mathbb{R}$, $b > 0$ and two events with probability $1/2$ each
- Let $w = (a, a+b)$, $f = (a, a-b)$ and $g = (a-b, a)$
- f is full insurance for w; $f \overset{\mathrm{d}}{=} g$
- Propension to full insurance implies

$$\mathbb{E}[u(w+f)] \geq \mathbb{E}[u(w+g)]$$

which is

$$u(2a) \geq \frac{1}{2}u(2a-b) + \frac{1}{2}u(2a+b)$$

- Since a, b are arbitrary this implies concavity of u
- $\succsim$ is risk averse

Risk-insurance equivalence

Theorem 3

The following properties are equivalent for a continuous risk preference:

(i) *strong risk aversion;*

(ii) *propension to proportional insurance;*

(iii) *propension to deductible-limit insurance.*

- (i) $\Rightarrow$ (ii) and (iii) in the literature Lorentz'53 AMM

  see Tchen'80 AOP; Rüschendorf'80 PTRF; Puccetti/W.'15 STS

More insurances

Given any initial wealth w, a random payoff f is:

(iv) an indemnity-schedule insurance for w, written $f \in \mathcal{I}^{\mathrm{is}}(w)$, when

$$f = I(-w)$$

for some real-valued (weakly) increasing map I;

(v) a contingency-schedule insurance for w, written $f \in \mathcal{I}^{\mathrm{cs}}(w)$, when

$$-w(s) > -w(s') \implies f(s) \geq f(s')$$

for almost all states s and s'. counter-monotonicity

Relation

$$\mathcal{I}^{\mathrm{pr}}(w) \cup \mathcal{I}^{\mathrm{dl}}(w) \subset \mathcal{I}^{\mathrm{is}}(w) \subset \mathcal{I}^{\mathrm{cs}}(w)$$

More insurances

(vi) Given any initial wealth w, a random payoff f is a better hedge for w than a random payoff g, written $f \geq_w g$, when $f \stackrel{d}{=} g$ and

$$P\left(f \leq t;\ w \leq l\right) \leq P\left(g \leq t;\ w \leq l\right)$$

for all payouts $t \in \mathbb{R}$ and wealth levels $l \in \mathbb{R}$.

- Copulas are ordered
- Equivalent condition:

$$P\left(f \leq t \mid w \leq l\right) \leq P\left(g \leq t \mid w \leq l\right)$$

More insurances

A risk preference $\succsim$ is:

(iv) propense to indemnity-schedule insurance when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\text{is}}(w) \implies w + f \succsim w + g;$$

(v) propense to contingency-schedule insurance when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\text{cs}}(w) \implies w + f \succsim w + g;$$

(vi) propense to hedging when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \geq_w g \implies w + f \succsim w + g.$$

More insurances

Theorem 4

The following conditions are equivalent for a continuous risk preference:

(i) *strong risk aversion;*

(ii) *propension to proportional insurance;*

(iii) *propension to deductible-limit insurance;*

(iv) *propension to indemnity-schedule insurance;*

(v) *propension to contingency-schedule insurance;*

(vi) *propension to hedging.*

Comparative attitudes

A risk preference $\succsim$ is secular when, for all $f, g \in \mathcal{F}$, there exists $\rho \in \mathbb{R}$, denoted by $\rho(f, g)$, such that

$$g \sim f - \rho$$

- Consider two agents Ann (A) and Bob (B) with ρ_{A} and ρ_{B}
- B is weakly more risk averse than A when Yaari'69 JET

$$f = \mathbb{E}[g] \implies \rho_{\mathrm{B}}(g, f) \geq \rho_{\mathrm{A}}(g, f)$$

- B is strongly more risk averse than A when Ross'81 ECMA

$$f \geq_{\mathrm{cv}} g \implies \rho_{\mathrm{B}}(g, f) \geq \rho_{\mathrm{A}}(g, f)$$

Comparative attitudes

Let $\succsim_{\mathrm{A}}$ and $\succsim_{\mathrm{B}}$ be monotone and secular risk preferences

- B is more propense to full insurance than A when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \in \mathcal{I}^{\mathrm{fi}}(w) \implies \rho_{\mathrm{B}}(w+g, w+f) \geq \rho_{\mathrm{A}}(w+g, w+f)$$

- Partial insurance: $\mathcal{I}^{\mathrm{fi}}$ is replaced by other sets of insurance
- B is more propense to hedging than A when, for all $w, f, g \in \mathcal{F}$ with $g \overset{d}{=} f$,

$$f \geq_w g \implies \rho_{\mathrm{B}}(w+g, w+f) \geq \rho_{\mathrm{A}}(w+g, w+f)$$

Summary

ABSOLUTE ATTITUDES

Arrow'63; Pratt'64		This paper
Weak risk aversion		**Propensity for full insurance**
$f = \mathbb{E}[g] \implies f \succsim g$	$\iff$	$f \overset{d}{=} g;\ f \in \mathcal{I}^{\mathrm{fi}}(w) \implies w + f \succsim w + g$
risk elimination is preferred		full insurance acquisition is preferred
$\Uparrow$		$\Uparrow$
Rothschild/Stiglitz'70		This paper
Strong risk aversion		**Propensity for partial insurance**
$f \geq_{\mathrm{cv}} g \implies f \succsim g$	$\iff$	$f \overset{d}{=} g;\ f \in \mathcal{I}^{\mathrm{pi}}(w) \implies w + f \succsim w + g$
risk reduction is preferred		partial insurance acquisition is preferred

Figure: Summary of absolute attitudes, where superscript pi is any one of $\mathrm{dl}, \mathrm{pr}, \mathrm{is}, \mathrm{cs}$

Summary

COMPARATIVE ATTITUDES

Yaari'69		This paper
Weakly more risk averse	$\Longleftrightarrow$	**More propense for full insurance**
$f = \mathbb{E}[g] \implies \rho_{\mathrm{B}}(g,f) \ge \rho_{\mathrm{A}}(g,f)$		$f \overset{d}{=} g;\ f \in \mathcal{I}^{\mathrm{is}}(w) \implies \rho_{\mathrm{B}}(w+f) \ge \rho_{\mathrm{A}}(w+g)$
risk elimination is more preferred		full insurance acquisition is more preferred
$\Uparrow$		$\Uparrow$
Ross'81		This paper
Strongly more risk averse	$\Longleftrightarrow$	**More propense for partial insurance**
$f \ge_{\mathrm{cv}} g \implies \rho_{\mathrm{B}}(g,f) \ge \rho_{\mathrm{A}}(g,f)$		$f \overset{d}{=} g;\ f \in \mathcal{I}^{\mathrm{pi}}(w) \implies \rho_{\mathrm{B}}(w+f) \ge \rho_{\mathrm{A}}(w+g)$
risk reduction is more preferred		partial insurance acquisition is more preferred

Figure: Summary of comparative attitudes, where superscript pi is any one of $\mathrm{dl}, \mathrm{pr}, \mathrm{is}, \mathrm{cs}$

Choice under dependence

Choice under dependence

Definition 1

Two acts f and g are comonotonic, written $f /\!/ g$, when

$$(f(s) - f(s'))\,(g(s) - g(s')) \geq 0$$

for all states s and s'. When $\leq$ is in place of $\geq$, we say that the two acts are counter-monotonic, written $f \backslash\!\backslash g$.

- Comonotonicity $\Longrightarrow$ no hedge
- Counter-monotonicity $\Longrightarrow$ maximum hedge

Choice under dependence

A set $\mathcal{D} \subseteq \mathcal{F}^2$ is dependence shell if it satisfies

$$(f, g) \in \mathcal{D} \text{ and } (f', g') \stackrel{\mathrm{d}}{=} (f, g) \implies (f', g') \in \mathcal{D}$$

- $\mathcal{D}$ describes a binary relation on joint distributions
- $\mathcal{D}$ is rich if for any (f, h), there exists g such that $f \stackrel{\mathrm{d}}{=} g$ and $(g, h) \in \mathcal{D}$

The following dependence shells are rich:

(i) $\mathcal{D}_{\mathrm{CM}} = \{(f, g) \in \mathcal{F}^2 : f /\!\!/ g\}$ (comonotonicity)

(ii) $\mathcal{D}_{\mathrm{CT}} = \{(f, g) \in \mathcal{F}^2 : f \backslash\!\!\backslash g\}$ (counter-comonotonicity)

(iii) $\mathcal{D}_{\mathrm{AL}} = \mathcal{F}^2$ (all)

Choice under dependence

Richness depends on the probability space:[1]

(iv) $\mathcal{D}_{\rm IN} = \{(f,g) \in \mathcal{F}^2 : f \perp g\}$ (independence)

Not rich:

(v) $\mathcal{D}_{\rm PL} = \{(f,g) \in \mathcal{F}^2 : f = ag + b \text{ for some } a > 0 \text{ and } b \in \mathbb{R}\}$ (positive linear dependence)

(vi) $\mathcal{D}_{\rm NL} = \{(f,g) \in \mathcal{F}^2 : f = ag + b \text{ for some } a < 0 \text{ and } b \in \mathbb{R}\}$ (negative linear dependence)

(vii) $\mathcal{D}_{\rm CS} = \{(f,g) \in \mathcal{F}^2 : f + g = \mathbb{E}(f+g)\}$ (constant sum)

- $\mathcal{D}_{\rm CS}$ is also called JM dependence — Wang/W.'16 MOR

[1] In an atomless probability space, richness of $\mathcal{D}_{\rm IN}$ means that for all $f \in \mathcal{F}$ there exists a continuously distributed random variable independent of f.

Choice under dependence

Definition 2

Let $\mathcal{D}$ be a dependence shell. A preference $\succsim$ is $\mathcal{D}$-averse if for all acts f, g, w,

$$f \overset{\mathrm{d}}{=} g \text{ and } (g, w) \in \mathcal{D} \implies w + f \succsim w + g.$$

A preference $\succsim$ is $\mathcal{D}$-propense if for all acts f, g, w,

$$f \overset{\mathrm{d}}{=} g \text{ and } (f, w) \in \mathcal{D} \implies w + f \succsim w + g.$$

A preference $\succsim$ is $\mathcal{D}$-neutral if it is both $\mathcal{D}$-averse and $\mathcal{D}$-propense.

- Example: propension to full insurance is $\mathcal{D}_{\mathrm{CS}}$-propension

Characterizing risk neutrality

$\mathcal{D}_{\mathrm{AL}}$-neutrality:

$$f \overset{\mathrm{d}}{=} g \implies w + f \sim w + g.$$

Theorem 5

For a binary transitive relation $\succsim$, the following are equivalent:

(i) $\succsim$ *satisfies $\mathcal{D}_{\mathrm{AL}}$-neutrality;*

(ii) $\succsim$ *is risk neutral.*

- A fundamental connection between risk attitude and dependence

Some mathematics

Proof of Theorem 5.

- Taking $w = 0$ yields

$$f \overset{\mathrm{d}}{=} g \implies f \sim g$$

- For any $f \in L^1$, by Theorem 2,

$$f \overset{\mathrm{d}}{=} g - g' + \mathbb{E}(f) \sim g - g + \mathbb{E}(f) = \mathbb{E}(f)$$

- $f \sim \mathbb{E}(f)$

Another equivalence

Proposition 1

Let $\mathcal{F} = \mathcal{M}^\infty$ and P be nonatomic. The following conditions are equivalent for a monotone risk preference $\succsim$:

(i) *for all* $f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}[f] \geq \mathbb{E}[g]$;

(ii) *for all* $w, f, g \in \mathcal{F}$, (*this paper*)

$$f \geq_{\mathrm{fsd}} g \implies w + f \succsim w + g;$$

(iii) *for all* $w, f, g \in \mathcal{F}$, (*de Finetti'31*)

$$f \succsim g \implies w + f \succsim w + g;$$

(iv) $\succsim$ *is complete and* (*Pomatto/Strack/Tamuz'20 JPE*)

$$f \succ g \implies w + f >_{\mathrm{fsd}} w + g$$

for some $w \in \mathcal{F}$ *independent of both* f *and* g *(if possible).*

Choice under dependence and risk aversion

Theorem 6

For a continuous risk preference $\succsim$, the following conditions are equivalent.

(i) $\succsim$ *is* $\mathcal{D}_{\mathrm{CT}}$*-propense;*

(ii) $\succsim$ *is* $\mathcal{D}_{\mathrm{CM}}$*-averse;*

(iii) $\succsim$ *is* $\mathcal{D}_{\mathrm{NL}}$*-propense;*

(iv) $\succsim$ *is* $\mathcal{D}_{\mathrm{PL}}$*-averse;*

(v) $\succsim$ *is strongly risk averse.*

Choice under dependence and risk aversion

Example (EU).

- Suppose that $\succsim$ is EU with (measurable) utility function u
- Take $a \in \mathbb{R}$, $b > 0$ and two events with probability $1/2$ each
- Let $w = (a, a+b)$, $f = (a, a-b)$ and $g = (a-b, a)$
- $f \stackrel{\mathrm{d}}{=} g$; f, w counter-monotonic; g, w comonotonic
- either $\mathcal{D}_{\mathrm{CT}}$-propension or $\mathcal{D}_{\mathrm{CM}}$-aversion implies

$$\mathbb{E}[u(w+f)] \geq \mathbb{E}[u(w+g)]$$

which is

$$u(2a) \geq \frac{1}{2}u(2a-b) + \frac{1}{2}u(2a+b)$$

- Since a, b are arbitrary this implies concavity of u
- $\succsim$ is strongly risk averse

Choice under dependence and risk aversion

Risk aversion $\Longrightarrow$ $\mathcal{D}_{\mathrm{CM}}$-aversion/$\mathcal{D}_{\mathrm{CT}}$-propension (classic)

see Tchen'80 AOP; Rüschendorf'80 PTRF; Puccetti/W.'15 STS

Reverse direction (more important for us):

Ceteris paribus, risk aversion can be inferred by,

- ★ a demand for insurance, or
- ★ a dislike of gambling

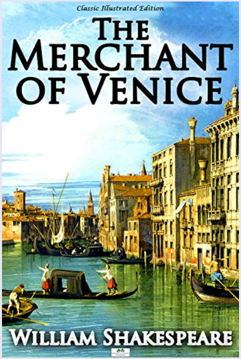
The chain

$$\mathcal{D}_{\mathrm{CS}} \subseteq \mathcal{D}_{\mathrm{CT}} \subseteq \mathcal{D}_{\mathrm{AL}}$$

corresponds to the following chain

weak risk aversion $\Longleftarrow$ strong risk aversion $\Longleftarrow$ risk neutrality

Is Antonio risk averse?



ACT 1, SCENE 1

ANTONIO:

Believe me, no. I thank my fortune for it,

My ventures are not in one bottom trusted,

Nor to one place; nor is my whole estate

Upon the fortune of this present year:

Therefore my merchandise makes me not sad.

(in response to SALARINO and SOLANIO)

$\Longrightarrow$ This is a choice under dependence

Is Antonio risk averse?

- Suppose that Antonio has two sets of commodities to deliver
- The first has payoff h
- The second has payoff f if it is on another boat
- The second has payoff g if it is on the same boat
- Two boats have the same subjective probability to return
- $f \overset{\mathrm{d}}{=} g$ and g, h comonotonic

Antonio says that commodities

not in one boat makes him not sad

$\Longrightarrow$ in one boat makes him sad

$\Longrightarrow$ $h + f \succsim h + g$

$\Longrightarrow$ Antonio is risk averse!

Is Antonio risk averse?

- Later Antonio takes a gamble with Shylock, but there was no comparable alternative presented $\Longrightarrow$ not a choice under dependence

ACT 1, SCENE 3

ANTONIO: *Come on: in this there can be no dismay;*

My ships come home a month before the day.

Risk measures

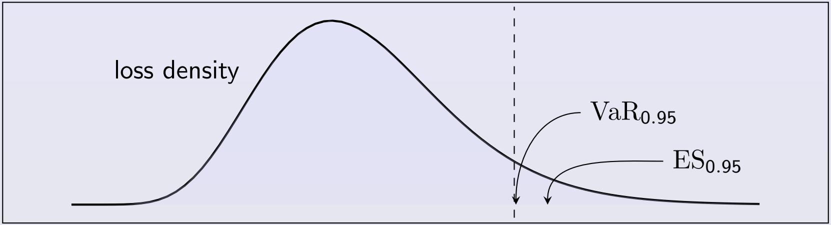
Risk measures

- Fix an atomless probability space $(S, \Sigma, \mathbb{P})$
- $\mathcal{X}$: the space of bounded random variables, representing losses
- A preference $\succsim$ is represented by a risk measure $\rho : \mathcal{X} \to \mathbb{R}$

$$X \succsim Y \iff \rho(X) \le \rho(Y)$$

- $\rho(X)$ is the amount of regulatory capital for a risk model X

VaR and ES



Value-at-Risk (VaR), $p \in (0,1)$

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

(left-quantile)

Expected Shortfall (ES), $p \in (0,1)$

$\mathrm{ES}_p : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR)

Some recent work on VaR and ES

- Axiomatic characterizations
  - VaR: Kou/Peng' 16 OR; He/Peng' 18 OR; Liu/W.'21 MOR
  - ES: W./Zitikis'21 MS; Embrechts/Mao/Wang/W.'21 MF
- Risk sharing
  - Embrechts/Liu/W.'18 OR; Embrechts/Liu/Mao/W.'20 MP
- Robustness
  - Embrechts/Wang/W.15 FS; Emberchts/Schied/W.'22 OR
- Calibrating levels between VaR and ES
  - Li/W.'23 JE
- Forecasting and backtesting
  - Fissler/Ziegel'16 AOS; Nolde/Ziegel'17 AOAS; Du/Escanciano'17 MS

Basic axioms

Basic axioms

M. (Monotonicity) $\rho(X) \le \rho(Y)$ whenever $X \le Y$.

TI. (Translation invariance) $\rho(X+m) = \rho(X) + m$ for $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

PH. (Positive homogeneity) $\rho(\lambda X) = \lambda \rho(X)$ for $X \in \mathcal{X}$ and $\lambda > 0$.

LI. (Law-invariance) $\rho(X) = \rho(Y)$ whenever $X \stackrel{\mathrm{d}}{=} Y$.

P. (Prudence) $\liminf_n \rho(\xi_n) \ge \rho(X)$ whenever $\xi_n \to X$.

- **M** and **TI**: monetary risk measures — Föllmer/Schied'02 FS
- **P**: the loss is modeled truthfully (e.g., consistent estimators) $\Longrightarrow$ estimated risk $\ge$ true risk asymptotically — W./Zitikis'21 MS
- For $p \in (0,1)$, both ES_p and VaR_p satisfy all above

Choice under dependence

Choice under dependence ($\mathcal{D}$-aversion):

$$X + Y \succsim X + Z, \qquad \text{with } Y \overset{\mathrm{d}}{=} Z$$

or, equivalently
$$\rho(X+Y) \le \rho(X+Z), \qquad \text{with } Y \overset{\mathrm{d}}{=} Z$$

for (X, Z) in some dependence shell $\mathcal{D}$ (undesirable)

How do we formulate undesirable dependence for portfolio risks?

- No condition on dependence $\Longrightarrow$ the mean — Theorem 5
- Comonotonicity $\Longrightarrow$ risk aversion — Theorem 6; Mao/W.'20 SIFIN
- Something less restrictive than comonotonicity?

Concentrated risks

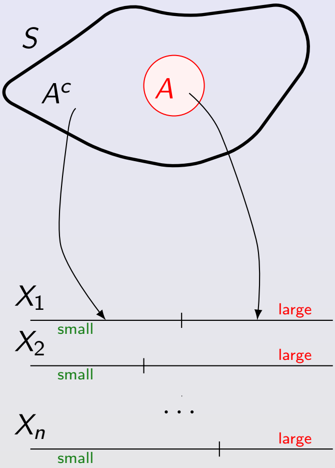
Definition 3 (Tail events)

A tail event of X is $A \in \Sigma$ such that

a) $0 < \mathbb{P}(A) < 1$

b) $X(\omega) \geq X(\omega')$

for a.e. all $\omega \in A$ and $\omega' \in A^c$



Undesirable dependence

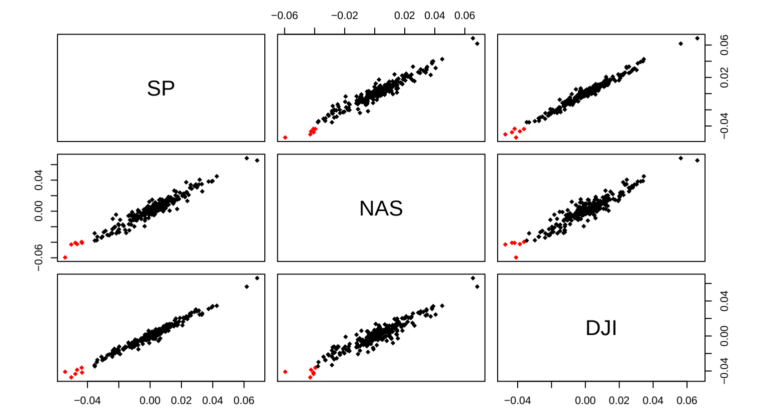
concentrated portfolio $\Longleftrightarrow$

severe losses occur simultaneously

on a stress event specified by the

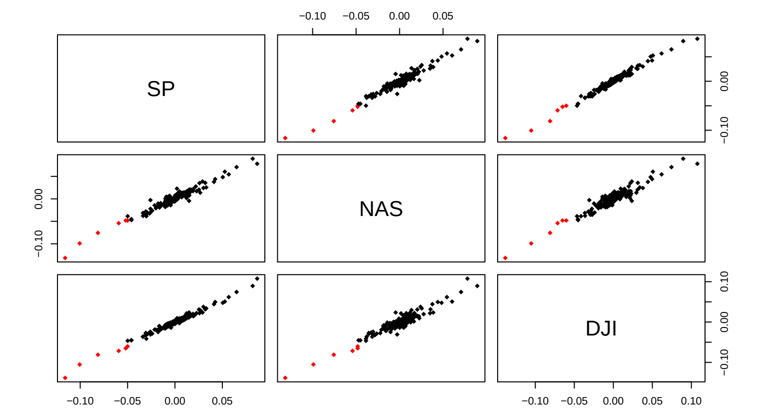
regulator

Risk concentration in 2009



S&P 500, NASDAQ and Dow Jones daily returns, Jan 2, 2009 - Dec 31, 2009

Risk concentration in 2019 - 2020



S&P 500, NASDAQ and Dow Jones daily returns, Jul 1, 2019 - Jun 30, 2020

Choice under dependence and ES

Concentration aversion

CA. (Concentration aversion) There exists an event $A \in \Sigma$ with $\mathbb{P}(A) \in (0,1)$ such that $\rho(X+Y) \le \rho(X+Z)$ if $Y \overset{\mathrm{d}}{=} Z$ and X and Z share the tail event A.

(non-concentrated) $X + Y \succsim X + Z$ (concentrated) with $Y \overset{\mathrm{d}}{=} Z$

Theorem 7 (Han/Wang/W./Wu'23 MF)

A functional $\rho : \mathcal{X} \to \mathbb{R}$ with $\rho(0) = 0$ satisfies Axioms **M**, **LI**, **TI**, **P** *and* **CA** *if and only if it is ES_p for some $p \in (0,1)$.*

- ρ satisfies **M**, **LI** and **CA** $\iff \rho = f(\mathrm{ES}_p, \mathbb{E})$ for increasing f

Expectiles

For $\alpha \in (0,1)$ and $X \in \mathcal{X}$, the α-expectile $e_\alpha(X)$ is the unique number y such that

$$\alpha \mathbb{E}\left[(X-y)_+\right] = (1-\alpha)\mathbb{E}\left[(y-X)_+\right]$$

Expectiles are

- introduced in asymmetric least squares — Newey/Powell'87 ECMA

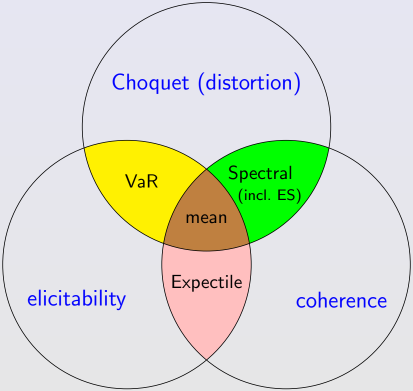
$$e_\alpha(X) = \arg\min_{y \in \mathbb{R}} \mathbb{E}\left[\alpha(X-y)_+^2 + (1-\alpha)(y-X)_+^2\right]$$

- coherent if $\alpha \geq 1/2$ — Bellini/Klar/Müller/Rosazza Gianin'14 IME
- elicitable — Ziegel'16 MF
- the mean if $\alpha = 1/2$

The risk measures diagram

Ch. (Choquet) **M** + **LI** + **TI** + comonotonic additivity

Co. (Coherence) **M** + **LI** + **TI** + convexity + **PH**



Choice under dependence and expectiles

Co-losses

Random variables X and Z are co-losses if $\{X>0\}=\{Z>0\}$.

Co-loss dependence aversion

CLA. (Co-loss aversion) $\rho(X+Y)\le\rho(X+Z)$ if $Y\overset{\mathrm{d}}{=}Z\sim 0$, and X and Z are co-losses.

(no co-loss) $X+Y \succsim X+Z$ (co-loss) with $Y\overset{\mathrm{d}}{=}Z\sim 0$

Theorem 8 (Bellini/Mao/W./Wu'23)

A functional $\rho:\mathcal{X}\to\mathbb{R}$ with $\rho(0)=0$ satisfies Axioms **M**, **TI**, **PH** *and* **CLA** *if and only if it is e_α for some $\alpha\in[1/2,1)$.*

Conclusion

Choices under dependence

- characterizes and explains
  - risk neutrality: $\mathcal{D}_{\mathrm{AL}}$-propension/aversion/neutrality
  - weak risk aversion: $\mathcal{D}_{\mathrm{CS}}$-propension
  - strong risk aversion: $\mathcal{D}_{\mathrm{CT}}$-propension/$\mathcal{D}_{\mathrm{CM}}$-aversion
- characterizes risk measures
  - arbitrary dependence: mean
  - concentration via tail events: ES
  - co-loss dependence: expectiles
- can be used to infer risk attitudes
- leads to new mathematics

Conclusion

Future directions on choice under dependence

- Other dependence concepts lead to different risk measures
  - VaR?
- Ambiguity preferences; multidimensional (systemic) risks
  - What is a notion of comparability similar to $\stackrel{\mathrm{d}}{=}$ for ambiguity?
- Can we model more delicate risk attitudes?
  - higher order, fractional order, loss aversion, wealth effect, ...
- How can we quantitatively infer risk aversion from observed portfolio strategies?
- What new notions of risk attitudes and risk measures can come out of this new framework?

Thank you

Thank you for your attention

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