Probability distortion, quantile maximization, and risk sharing

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June 2, 2022

Agenda

1. Background
2. Distributional transforms and probability distortions
3. Axiomatization of quantiles and quantile maximization
4. Quantile-based risk sharing

Based on

- Liu-Schied-W. Distributional transforms, probability distortions, and their applications. Mathematics of Operations Research, 2021
- Fadina-Liu-W. One axiom to rule them all: A minimalist axiomatization of quantiles. SSRN: 3944312, 2022
- Embrechts-Liu-W., Quantile-based risk sharing. Operations Research, 2018

and some work in progress

A Chinese phrase: Banmennongfu

Expected utility theory

- $\mathcal{M}$: a set of distributions on $\mathbb{R}$

The von Neumann-Morgenstein expected utility: $\mathcal{U}_u : \mathcal{M} \to \overline{\mathbb{R}}$,

$$\mathcal{U}_u(F) = \int_{\mathbb{R}} u(x)\mathrm{d}F(x)$$

- $u : \mathbb{R} \to \mathbb{R}$ is a utility function
- $\mathcal{U}_u$ is linear in the distribution function F
- Key axiom of independence: $\forall H \in \mathcal{M},\ \lambda \in (0,1]$,

$$F \succeq G \iff \lambda F + (1-\lambda)H \succeq \lambda G + (1-\lambda)H$$

- Axiomatized by Savage with subjective probability
- Challenged by e.g., the Allais and the Ellsberg paradoxes

Dual utility theory

The dual utility of Yaari (assuming h is continuous): $\mathcal{D}_h : \mathcal{M} \to \overline{\mathbb{R}}$,

$$\mathcal{D}_h(F) = \int_{\mathbb{R}} x \mathrm{d}(h \circ F)(x) = \int_0^1 F^{-1}(t) \mathrm{d}h(t)$$

- $h : [0,1] \to [0,1]$ is a distortion (or perception) function
- $\mathcal{D}_h$ is linear in the (left) quantile function F^{-1}
- Key axiom of dual independence: $\forall H \in \mathcal{M},\ \lambda \in (0,1]$,

$$F \succeq G \iff F \oplus_\lambda H \succeq G \oplus_\lambda H$$

  where $F \oplus_\lambda H$ has quantile function $\lambda F^{-1} + (1-\lambda) H^{-1}$
- Scalability: $X \succeq Y \iff aX \succeq aY$ for all $a > 0$

Rank-dependent utility theory

The rank dependent utility (RDU) of Quiggin $\mathcal{R}_{u,h} : \mathcal{M} \to \overline{\mathbb{R}}$:

$$\mathcal{R}_{u,h}(F) = \int_{\mathbb{R}} u(x)\mathrm{d}(h \circ F)(x)$$

- u is a utility function
- h is a distortion function
- The cumulative prospect theory of Kahneman-Tversky generalizes RDU

Risk measures

A law-based risk measure $\rho : \mathcal{M} \to \mathbb{R}$

- represents regulatory capital
- Artzner-Delbaen-Eber-Heath'99 MF, Follmer-Schied'02 FS, Frittelli-Rosazza Gianin'02 JBF

Key example: the Expected Shortfall (ES) for $p \in (0,1)$,

$$\mathrm{ES}_p(F) = \frac{1}{1-p}\int_p^1 F^{-1}(t)\mathrm{d}t$$

- $\mathrm{ES}_{0.975}$ is the standard market risk measure in Basel IV
- Axiomatized recently by W.-Zitikis'21 MS

Two-step procedure

- Expected utility

$$F \implies F_{u(X)} := F \circ u^{-1} \implies \mathbb{E}[F \circ u^{-1}] = \mathcal{U}_u(F)$$

- Dual utility

$$F \implies h \circ F \implies \mathbb{E}[h \circ F] = \mathcal{D}_h(F)$$

- RDU

$$F \implies h \circ F \circ u^{-1} \implies \mathbb{E}[h \circ F \circ u^{-1}] = \mathcal{R}_{u,h}(F)$$

- ES

$$F \implies F_p := (F - p)_+/(1 - p) \implies \mathbb{E}[F_p] = \mathrm{ES}_p(F)$$

◇ Step 1: transform F to another distribution F'

◇ Step 2: compute the mean of F'

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Distributional transforms

- $\mathcal{M}_0$: the set of all distributions on $\mathbb{R}$, represented by cdfs
- $\mathcal{M}_c$: compactly supported; $\mathcal{M} \subseteq \mathcal{M}_0$

Definition

A distributional transform T is a mapping from $\mathcal{M}$ to $\mathcal{M}_0$.

Definition

For a monotone function ϕ on $\mathbb{R}$, the utility transform (UT) generated by ϕ, denoted by $T^{[\phi]} : \mathcal{M} \to \mathcal{M}_0$, is defined as the distribution of $\phi(X)$ where $X \sim F$.

- Treating distributions as measures: $T^{[\phi]}(F) = F \circ \phi^{-1}$

Probability distortions

- $\mathcal{H}$: the set of all increasing $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$
- $[F]$: the right-continuous version of F

Definition

For $g \in \mathcal{H}$, the probability distortion (PD) generated by g, denoted by $T_g : \mathcal{M} \to \mathcal{M}_0$, is defined as $T_g(F) = [g \circ F]$ (if well-defined).

- If g is right-continuous, then $T_g(F) = g \circ F$

T_g satisfies many properties

- $\leq_{\mathrm{st}}$-monotone, constant-preserving, lower semi-continuous, ...
- $\leq_{\mathrm{icx}}$-monotone $\iff$ g is convex

Examples of distributional transforms

- Tail transform: $T(F) = (F-p)_+/(1-p)$ for some $p \in (0,1)$ [PD]
- Distorted power transform: $T(F) = F^\gamma$ for $\gamma > 0$ [PD]
- Scale-location transform: $T(F)$ is the distribution of $aX + b$, where $X \sim F$, for $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ [UT]
- Super-quantile transform: $T(F)$ has a quantile function given by $p \mapsto \mathrm{ES}_p(F)$
- Lorenz curve: $T(F) : x \mapsto \frac{\int_0^x F^{-1}(t)\mathrm{d}t}{\int_0^1 F^{-1}(t)\mathrm{d}t}$ on $[0,1]$
- Convolution transform: $T(F) : x \mapsto \int F(x-y)\, G(\mathrm{d}y)$ for a fixed $G \in \mathcal{M}_0$
- Weighted transform: $T(F)$ has density $\frac{w(x)f(x)}{\int_{\mathbb{R}} w(y)f(y)\mathrm{d}y}$ where f is the density of F

A characterization of probability distortions

- $\mathcal{G}^*$: the set of continuous and increasing functions on $\mathbb{R}$
- L and R stand for left- and right-continuous ones

Theorem

For a mapping $T : \mathcal{M}_c \to \mathcal{M}_c$,

(i) *T commutes with $T^{[\phi]}$ for all $\phi \in \mathcal{G}^*$ if and only if $T = T_g$ for some $g \in \mathcal{H}$;*

(ii) *T commutes with $T^{[\phi]}$ for all $\phi \in \mathcal{G}^L$ if and only if $T = T_g$ for some $g \in \mathcal{H}^R$.*

- Commuting with UT characterizes PD
- (Non-linear) scaling does not matter $\Rightarrow$ PD users

A characterization of utility transforms

Theorem

For a mapping $T : \mathcal{M}_c \to \mathcal{M}_c$,

(i) T *commutes with* T_g *for all* $g \in \mathcal{H}$ *if and only if* $T = T^{[\phi]}$ *for some* $\phi \in \mathcal{G}^*$.

(ii) T *commutes with* T_g *for all* $g \in \mathcal{H}^R$ *if and only if* $T = T^{[\phi]}$ *for some* $\phi \in \mathcal{G}^L$;

- Commuting with PD characterizes UT
- Probability reweighting does not matter $\Rightarrow$ UT users

RDU transforms

- $\mathcal{G}^{\diamond}_{\pm} = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi$ is strictly monotone, continuous, $\phi(\mathbb{R}) = \mathbb{R}\}$
- T is $\mathcal{G}$-semi-covariant if for each $\phi \in \mathcal{G}$, there exist $\psi, \eta \in \mathcal{G}$ such that $T^{[\phi]} \circ T = T \circ T^{[\psi]}$ and $T \circ T^{[\phi]} = T^{[\eta]} \circ T$

Theorem

A mapping $T : \mathcal{M}_c \to \mathcal{M}_c$ is $\mathcal{G}^*$-semi-covariant if and only if $T = T_g \circ T^{[\xi]}$ for some $g \in \mathcal{H}$ and $\xi \in \mathcal{G}^{\diamond}_{\pm}$.

- Fix $g \in \mathcal{H}$, $\xi \in \mathcal{G}^{\diamond}_{\pm}$ and $\phi \in \mathcal{G}^*$; $T = T_g \circ T^{[\xi]}$

$$T^{[\phi]} \circ T_g \circ T^{[\xi]} = T_g \circ T^{[\phi]} \circ T^{[\xi]} = T_g \circ T^{[\xi]} \circ T^{[\xi^{-1}]} \circ T^{[\phi]} \circ T^{[\xi]}$$

- Each (non-linear) scaling on input translates into a scaling on the output $\Rightarrow$ RDU transform users

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Axiomatization of quantiles

A mapping $\rho : \mathcal{M} \to \mathbb{R}$ is

- a left quantile if for some $\alpha \in (0, 1]$,

$$\rho(F) = F_L^{-1}(\alpha) := \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

- a right quantile if for some $p \in [0, 1)$,

$$\rho(F) = F_R^{-1}(\alpha) := \inf\{x \in \mathbb{R} : F(x) > \alpha\}$$

Axiomatic characterizations of quantiles (VaR) as law-based mappings

- Chambers'09 MF: ordinal-covariance + monotonicity + semi-continuity
- Kou-Peng'16 OR: elicitability + comonotonic-additivity + ...
- He-Peng'18 OR: surplus-invariance + positive homogeneity + ...
- Liu-W.'21 MOR: elicitability + tail-relevance + ...

Axiomatization of quantiles

Definition

For a mapping $\rho : \mathcal{M} \to \mathbb{R}$ and a set $\mathcal{G}$ of measurable functions, we say that ρ is $\mathcal{G}$-ordinal if $\rho \circ T^{[\phi]} = \phi \circ \rho$ for all $\phi \in \mathcal{G}$.

- $\mathcal{G}^\diamond = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi$ is strictly increasing and continuous$\}$

Theorem (Chambers'09 MF)

A mapping $\rho : \mathcal{M}_c \to \mathbb{R}$ is $\mathcal{G}^\diamond$-ordinal, lower semi-continuous, and increasing if and only if ρ is a left quantile.

- *Replacing "lower" by "upper" $\Longrightarrow$ right quantiles*

Axiomatization of quantiles

Theorem

A mapping $\rho : \mathcal{M} \to \mathbb{R}$ with $\mathcal{M}_c \subseteq \mathcal{M} \subseteq \mathcal{M}_0$ is $\mathcal{G}^*$-ordinal if and only if it is a left or right quantile.

- Only one axiom
- No continuity
- No monotonicity
- General domains
- Ordinality alone characterizes quantiles

Quantile intervals and medians

Other results

- $\mathcal{G}^L$-ordinality $\iff$ ρ is a left quantile
- $\mathcal{G}^R$-ordinality $\iff$ ρ is a right quantile

If $\mathcal{M}$ has only continuous quantile functions

- $\mathcal{G}^*_{\pm}$-ordinality $\iff$ ρ is the median

$\rho : \mathcal{M} \to \mathbb{I}$ where $\mathbb{I}$ is the set of closed intervals in $\mathbb{R}$

- $\mathcal{G}^*$-ordinality $\iff$ ρ is a quantile interval
- $\mathcal{G}^*_{\pm}$-ordinality $\iff$ ρ is an equal-tailed quantile interval
- minimal $\mathcal{G}^*$-ordinality $\iff$ ρ is a quantile singleton
- minimal $\mathcal{G}^*_{\pm}$-ordinality $\iff$ ρ is the median interval

Quantile maximization

- $\preceq$ is a law-based preference on $\mathcal{X}$ (bounded random variables)
- $\mathcal{G}$-invariance of $\preceq$:

$$X \preceq Y \implies \phi(X) \preceq \phi(Y) \text{ for all } \phi \in \mathcal{G}$$

Theorem

A law-based total preorder $\preceq$ on $\mathcal{X}$ with certainty equivalents is $\mathcal{G}^*$-invariant if and only if it is a quantile maximizer.

- A quantle maximizer: $X \preceq Y \iff \mathcal{R}(X) \preceq \mathcal{R}(Y)$ where $\mathcal{R} = \lambda Q^L_\alpha$ or λQ^R_α for some $\lambda \in \mathbb{R}$ and α below
- $Q^L_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\},\ \alpha \in (0, 1]$
- $Q^R_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) > \alpha\},\ \alpha \in [0, 1)$

Progress

Quantile-based risk sharing

Given a risk $X \in \mathcal{X}$ shared by n agents, the set of allocations is

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}$$

What is a "canonical form" of an optimal (sensible) allocation?

If we assume the preferences of the agents are "similar"

- $X_i = a_i X +$ side payments for some $\sum_{i=1}^n a_i = 1$?
- $X_i = \mathbb{1}_{A_i} X +$ side payments for some $\bigcup_{i=1}^n A_i = \Omega$?

Quantile-based risk sharing

Theorem

For quantile maximizers $Q^R_{\alpha_1}, \dots, Q^R_{\alpha_n}$ with $\alpha := \sum_{i=1}^n \alpha_i < 1$, a Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ of X is given by

$$X_i^* = (X - z)\mathbb{1}_{A_i} + c_i, \quad i = 1, \dots, n.$$

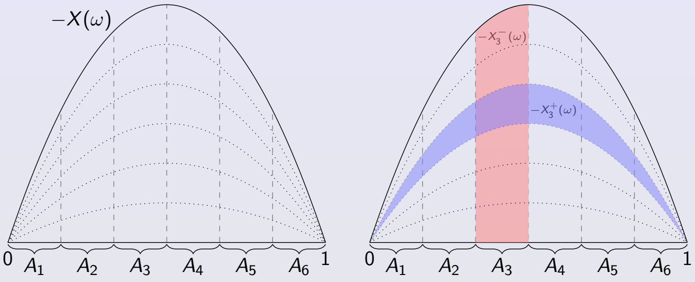
for some partition $(A_1, \dots, A_n)$ of Ω and $z, c_1, \dots, c_n \in \mathbb{R}$, where z is large enough.

- If $X \le 0$, an optimal allocation can be chosen as

$$X_i^* = X\mathbb{1}_{A_i}, \quad i = 1, \dots, n.$$

- No equilibrium condition: $\sum_{i=1}^n \alpha_i \ge 1$

Optimal allocations



- $\Omega = [0,1]$, $n = 6$, $X(\omega) = \omega^2 - \omega$
- A positively dependent allocation: $X_i^+ = X/n$, $i \in [n]$
  - the area between two dotted curves; utility maximizers
- A negatively dependent allocation: $X_i^- = X\mathbb{1}_{A_i}$, $i \in [n]$
  - the area between two dashed lines; quantile maximizers

Optimal allocations

quantile maximizers agent i: $Q^R_{\alpha_i}(X_i)$
vertical cut $(X\mathbb{1}_{A_1},\dots,X\mathbb{1}_{A_n})$ roulette negative dependence

utility maximizers agent i: $\mathbb{E}[u_i(X_i)]$
horizontal cut $(X/n,\dots,X/n)$ coinsurance positive dependence

Quantile inequalities

Assume $\alpha := \sum_{i=1}^n \alpha_i < 1$

- For any $(X_1, \dots, X_n) \in \mathcal{X}^n$,

$$\sum_{i=1}^n Q^R_{\alpha_i}(X_i) \le Q^R_\alpha\left(\sum_{i=1}^n X_i\right)$$

$$\sum_{i=1}^n Q^L_{1-\alpha_i}(X_i) \ge Q^L_{1-\alpha}\left(\sum_{i=1}^n X_i\right)$$

- For any $X \in \mathcal{X}$,

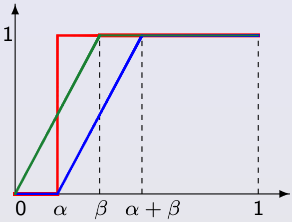
$$\max_{(X_1,\dots,X_n)\in\mathbb{A}_n(X)} \sum_{i=1}^n Q^R_{\alpha_i}(X_i) = Q^R_\alpha(X)$$

$$\min_{(X_1,\dots,X_n)\in\mathbb{A}_n(X)} \sum_{i=1}^n Q^L_{1-\alpha_i}(X_i) = Q^L_{1-\alpha}(X)$$

Quantile-based risk sharing

The allocation is optimal in settings of

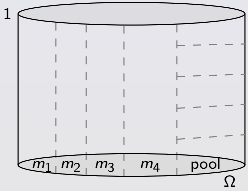
- competitive equilibria
- RVaR (including VaR and ES)
- heterogenous beliefs (Embrechts-Liu-Mao-W.'20 MP)



Distortion functions of VaR_α (red), ES_β (green) and $\mathrm{RVaR}_{\alpha,\beta}$ (blue)

Block rewards of Bitcoin mining

- Each Bitcoin miner $i \in \{1, \ldots, n\}$ has a computational contribution x_i mining a block
  - $p_i = x_i / \sum_{j=1}^n x_j$: the percentage contribution
- The Bitcoin reward protocol
  - randomly assigns the block to miner i with probability p_i
  - $\tilde{x}_i = 1_{A_i}$ with $\mathbb{P}(A_i) = p_i$
  - axiomatized by Leshno-Strack'20 AER-I
- Mining pool: proportional reward within a group
- Individual miners vs mining pools
  - quantile maximizer vs utility maximizer
  - decentralization vs centralization



Thank you

Thank you for your kind attention

Based on joint work with

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Working papers series on the theory of risk measures
http://sas.uwaterloo.ca/~wang/pages/WPS1.html