Diversification quotients: Quantifying diversification via risk measures

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Agenda

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Diversification

Portfolio diversification

- Markowitz (mean-variance analysis)
- CAPM (diversification $\Rightarrow$ removing idiosyncratic risk)
- Is diversification well-defined?
- Is diversification desirable? In what sense?

Main question: How do we measure diversification?

As least two approaches

- Heuristic: number of different investments
- Quantitative: formal reasoning
  - via risk reduction or utility improvement

Diversification indices

Quantitative setup

- $\mathcal{X}$: a convex cone of random variables, e.g., L^1
- One-period portfolio loss/payoff vector: $\mathbf{X} \in \mathcal{X}^n$
- Diversification index: $D : \mathcal{X}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$
  - Convention: smaller D represents better diversification
  - Always write $\mathbf{X} = (X_1, \dots, X_n)$

Examples: diversification ratios (DR) with $0/0 = 0$

$$\mathrm{DR}^{\mathrm{SD}}(\mathbf{X}) = \frac{\mathrm{SD}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{SD}(X_i)} \quad \text{and} \quad \mathrm{DR}^{\mathrm{var}}(\mathbf{X}) = \frac{\mathrm{var}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{var}(X_i)}$$

Diversification indices

$\mathrm{DR}^{\mathrm{SD}}$ and $\mathrm{DR}^{\mathrm{var}}$ satisfy three natural properties

[+] Non-negativity: $D(\mathbf{X}) \geq 0$ for all $\mathbf{X} \in \mathcal{X}^n$

- with $D = 0$ being the most diversified

[LI] Location invariance: $D(\mathbf{X} + \mathbf{c}) = D(\mathbf{X})$ for all $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathcal{X}^n$

- injecting risk-free payoff to each component does not affect D
- changing initial price of each component does not affect D

[SI] Scale invariance: $D(\lambda \mathbf{X}) = D(\mathbf{X})$ for all $\lambda > 0$ and $\mathbf{X} \in \mathcal{X}^n$

- rescaling of a portfolio does not affect D
- the counting unit or currency (non-random) does not affect D

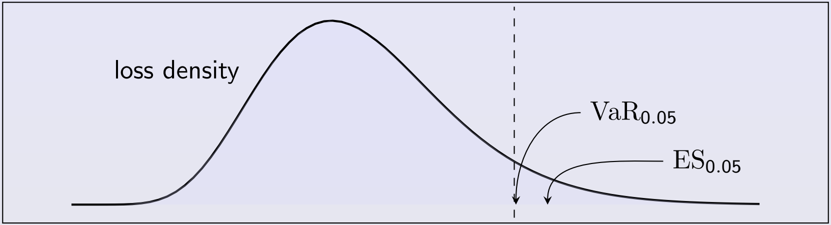
Generally not convex

Diversification indices

- SD and var are simple but coarse measures of risk
- By internal needs or regulation, risk should be assessed by risk measures $\tau : \mathcal{X} \to \mathbb{R}$
  - regulatory capital calculation, capital allocation, performance analysis, optimization, ...
  - VaR and ES (CVaR) are popular in banking and insurance regulatory frameworks, such as Basel III/IV and Solvency II
  - monetary/convex/coherent risk measures (Artzner/Delbaen/Eber/Heath'99; Follmer/Schied'16)

What is a suitable diversification index based on risk measures?

VaR and ES



Value-at-Risk (VaR), $\alpha \in (0,1)$

$\mathrm{VaR}_\alpha : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_\alpha(X) = q_\alpha(X)$$
$$= \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge 1 - \alpha\}$$

(left-quantile)

Expected Shortfall (ES), $\alpha \in (0,1)$

$\mathrm{ES}_\alpha : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\mathrm{d}\beta$$

(also: TVaR/CVaR/AVaR)

Diversification ratios based on risk measures

Some candidates (Tasche'07; McNeil/Frey/Embrechts'15)

$$\mathrm{DR}^{\tau}(\mathbf{X}) = \frac{\tau\left(\sum_{i=1}^{n} X_i\right)}{\sum_{i=1}^{n} \tau(X_i)} \quad \text{and} \quad \mathrm{DB}^{\tau}(\mathbf{X}) = \sum_{i=1}^{n} \tau(X_i) - \tau\left(\sum_{i=1}^{n} X_i\right)$$

D	[+]	[LI]	[SI]
DR^{τ}	No	No	τ pos. hom.
DB^{τ}	τ subadditive	τ cons. add.	No

- DR is impossible to interpret if "negative over negative"
- awkward for optimization
- wrong incentives in some simple models ($\Rightarrow$ next slide)
- $\Longrightarrow$ a new index is needed if risk measures are used

Diversification ratios based on risk measures

Consider three models

1. iid standard normal: $\mathbf{Z} = (Z_1, \dots, Z_n)$
2. iid shock: $\mathbf{Y}' = (\xi_1 Z_1, \dots, \xi_n Z_n)$ where $\xi_1, \dots, \xi_n$ are iid heavy-tailed shocks independent of $\mathbf{Z}$
3. common shock: $\mathbf{Y} = (\xi Z_1, \dots, \xi Z_n)$ where $\xi \overset{\mathrm{d}}{=} \xi_1$ is a heavy-tailed shock independent of $\mathbf{Z}$

Diversification ratios based on risk measures

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Intuitive relation on diversification (smaller $\Rightarrow$ better):

$$\text{Model } 1 \le \text{Model } 2 < \text{Model } 3$$

- If $\xi^2 \sim \mathrm{ig}(\nu/2, \nu/2)$, then $\mathbf{Y}' \sim \mathrm{it}_n(\nu)$ and $\mathbf{Y} \sim \mathrm{t}(\nu, \mathbf{0}, I_n)$

Diversification ratios based on risk measures

Table: DR and DB, where $\alpha = 0.05$ and $n = 10$

D	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	$\mathrm{DR}^{\mathrm{var}}$	$\mathrm{DB}^{\mathrm{VaR}_\alpha}$
$\mathbf{Y}' \sim \mathrm{it}_n(3)$	0.3568	0.3058	0.3162	1	15.131
$\mathbf{Y} \sim \mathrm{t}(3, \mathbf{0}, I_n)$	0.3162	0.3162	0.3162	1	16.092
$D(\mathbf{Y}')/D(\mathbf{Y})$	1.1284	0.9671	1	1	0.9404
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	0.3162	0.3162	0.3162	1	11.247
$D(\mathbf{Z})/D(\mathbf{Y})$	1	1	1	1	0.6989

Diversification indices based on risk measures

Question: Can we find a diversification index that is

- based on a specified risk measure (e.g., VaR or ES)
- satisfying the three natural properties [+], [SI] and [LI]
- consistent with common portfolio dependence structures
- natural to interpret
- able to capture heavy tails and common shocks
- convenient to compute and optimize for portfolio selection?

Progress

Risk measures

Standard properties for a risk measure $\tau : \mathcal{X} \to \mathbb{R}$

[CA]$_m$ Constant additivity with $m \in \mathbb{R}$: $\tau(X + c) = \tau(X) + mc$ for all $c \in \mathbb{R}$ and $X \in \mathcal{X}$

[PH]$_\gamma$ Positive homogeneity with $\gamma \in \mathbb{R}$: $\tau(\lambda X) = \lambda^\gamma \tau(X)$ for all $\lambda \in (0, \infty)$ and $X \in \mathcal{X}$

[SA] Subadditivity: $\tau(X + Y) \leq \tau(X) + \tau(Y)$ for all $X, Y \in \mathcal{X}$

"[PH] holds" means "[PH]$_\gamma$ holds for some $\gamma \in \mathbb{R}$"; same for [CA]

- Coherent risk measures (incl. ES) satisfy [CA]$_1$, [PH]$_1$ & [SA]
- VaR_α satisfies [CA]$_1$ & [PH]$_1$
- var satisfies [CA]$_0$ & [PH]$_2$
- SD satisfies [CA]$_0$, [PH]$_1$ & [SA]

An impossibility theorem

- D is τ-based if $D(\mathbf{X}) = R\left(\tau\left(\sum_{i=1}^n X_i\right), \tau(X_1), \dots, \tau(X_n)\right)$ for some $R : \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$
- DR^τ is of this type

Theorem 1

Fix $n \geq 1$. Suppose that a risk measure τ satisfies [PH] *and* $[\mathrm{CA}]_m$ *with $m \neq 0$. A τ-based diversification index D satisfies* [+], [LI] *and* [SI] *if any only if for all $\mathbf{X} \in \mathcal{X}^n$,*

$$D(\mathbf{X}) = C_1 \mathbb{1}_{\{d<0\}} + C_2 \mathbb{1}_{\{d=0\}} + C_3 \mathbb{1}_{\{d>0\}},$$

where $d = \tau\left(\sum_{i=1}^n X_i\right) - \sum_{i=1}^n \tau(X_i)$ for some $C_1, C_2, C_3 \in [0, \infty]$.

- In this case D is degenerate

Diversification quotients

- $\rho = (\rho_\alpha)_{\alpha \in I}$ decreasing in $\alpha \in I = (0, \overline{\alpha})$, $\overline{\alpha} \in (0, \infty]$
- Examples: $(\mathrm{VaR}_\alpha)_{\alpha \in (0,1)}$, $(\mathrm{ES}_\alpha)_{\alpha \in (0,1)}$

Definition 1 (Diversification quotients)

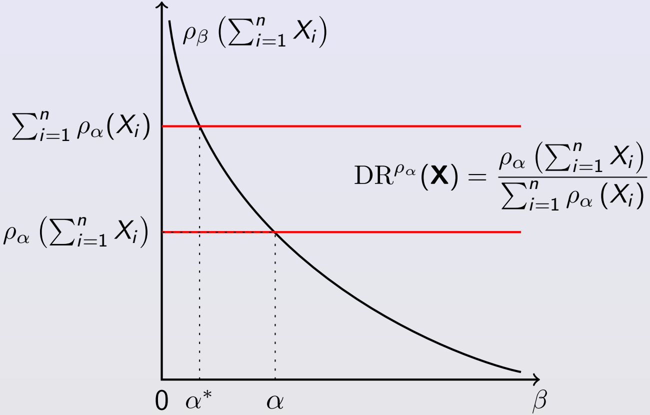
For $\mathbf{X} \in \mathcal{X}^n$, the diversification quotient based on the class ρ at level $\alpha \in I$ is defined by $\mathrm{DQ}^\rho_\alpha(\mathbf{X}) = \alpha^*/\alpha$, where

$$\alpha^* = \inf \left\{ \beta \in I : \rho_\beta \left(\sum_{i=1}^n X_i \right) \le \sum_{i=1}^n \rho_\alpha(X_i) \right\},$$

with the convention $\inf(\varnothing) = \overline{\alpha}$.

- $\mathrm{DQ}^\rho_\alpha(\mathbf{c}) = 0$ for $\mathbf{c} \in \mathbb{R}^n$ if $\rho_\beta(c) = c$ for $c \in \mathbb{R}$ and $\beta \in I$
- $\mathrm{DQ}^\rho_\alpha = \mathrm{DQ}^{\rho'}_1$ by re-parametrization via $\rho'_\beta = \rho_{\alpha\beta}$

Diversification quotients



$$\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \alpha^*/\alpha$$

Another DQ

Properties of DQ

Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a decreasing class of risk measures

- For each $\alpha \in I$, $\mathrm{DQ}^{\rho}_{\alpha}$ satisfies [+]
- If ρ satisfies [PH], then $\mathrm{DQ}^{\rho}_{\alpha}$ satisfies [SI] for each $\alpha \in I$
- If ρ satisfies [CA], then $\mathrm{DQ}^{\rho}_{\alpha}$ satisfies [LI] for each $\alpha \in I$
- For each $\alpha \in I$, if ρ_α satisfies [SA], then $\mathrm{DQ}^{\rho}_{\alpha}$ takes value in $[0, 1]$
- Monotonicity can be embedded into DQ
    - risk diversification vs variability diversification
    - monotonicity of a risk measure τ: $\tau(X) \le \tau(Y)$ if $X \le Y$
    - monetary risk measure: monotone and $[\mathrm{CA}]_1$

Connection to acceptability index

Acceptability index (Cherny/Madan'09):

$$\mathrm{AI}^{\rho}(X) = \inf \{\beta \in I : \rho_{\beta}(X) \le 0\}$$

If ρ satisfies $[\mathrm{CA}]_1$ then

$$\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \mathrm{AI}^{\rho}\left(\sum_{i=1}^{n}(X_i - \rho_{\alpha}(X_i))\right)$$

Interpreting DQ from portfolio risks

Three diversification situations:

(i) there is no insolvency risk with pooled individual capital, i.e., $\sum_{i=1}^n X_i \le \sum_{i=1}^n \tau(X_i)$ a.s.

(ii) diversification benefit exists, i.e., $\tau\left(\sum_{i=1}^n X_i\right) < \sum_{i=1}^n \tau(X_i)$

(iii) the portfolio relies on a single asset, i.e., $\mathbf{X} = (\lambda_1 X, \dots, \lambda_n X)$ for some $X \in \mathcal{X}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+ := [0, \infty)$

Properties of ρ:

- ρ is non-flat from the left at (α, X) if $\rho_\beta(X) > \rho_\alpha(X)$ for all $\beta \in (0, \alpha)$.
- ρ is left continuous at (α, X) if $\alpha \mapsto \rho_\alpha(X)$ is left continuous
- $\rho_0 = \lim_{\alpha \downarrow 0} \rho_\alpha$

Interpreting DQ from portfolio risks

Theorem 2

For given $\mathbf{X} \in \mathcal{X}^n$ *and* $\alpha \in I$, *if* ρ *is left continuous and non-flat from the left at* $(\alpha, \sum_{i=1}^n X_i)$, *the following hold.*

(i) *Suppose that* $\rho_0 \leq \text{ess-sup}$. *If for* ρ_α *there is no insolvency risk with pooled individual capital, then* $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 0$. *The converse holds true if* $\rho_0 = \text{ess-sup}$.

(ii) *Diversification benefit exists if and only if* $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) < 1$.

(iii) *If* ρ_α *satisfies* $[\mathrm{PH}]_1$ *and* $\mathbf{X}$ *relies on a single asset, then* $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 1$.

(iv) *If* ρ_α *is comonotonic-additive and* $\mathbf{X}$ *is comonotonic, then* $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 1$.

Axiomatic foundation

Setting

- $\mathcal{X} = L^\infty$ and $\tau : \mathcal{X} \to \mathbb{R}$ is a risk measure
- A risk measure is scalable if it satisfies $[\mathrm{PH}]_1$
- τ-marginal equivalence ($\mathbf{X} \overset{\tau}{\sim} \mathbf{Y}$): $\tau(X_i) = \tau(Y_i)$ for each i
- $\mathbf{0}$ is the n-vector of zeros

Axioms

$[\mathrm{M}]_\tau$ Monotonicity under τ-marginal equivalence: $D(\mathbf{X}) \le D(\mathbf{Y})$ for $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^n$ satisfying $\mathbf{X} \overset{\tau}{\sim} \mathbf{Y}$ and $\sum_{i=1}^n X_i \le \sum_{i=1}^n Y_i$

[C] Continuity: The set $\{\sum_{i=1}^n X_i : \mathbf{X} \in \mathcal{X}^n,\ D(\mathbf{X}) \le \beta,\ \mathbf{X} \overset{\tau}{\sim} \mathbf{0}\}$ is L^∞-closed and non-empty for each $\beta \ge 0$

Axiomatic foundation

For a decreasing family $\rho = (\rho_\alpha)_{\alpha \in I}$ and a risk measure τ, define

$$\mathrm{DQ}_\tau^\rho(\mathbf{X}) = \inf\left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \tau(X_i)\right\}$$

If $\tau = \rho_\alpha$, then $\mathrm{DQ}_\tau^\rho = \alpha \mathrm{DQ}_\alpha^\rho$

Theorem 3

Let $\tau : \mathcal{X} \to \mathbb{R}$ be a monetary scalable risk measure. A diversification index $D : \mathcal{X}^n \to \overline{\mathbb{R}}$ satisfies [+], [LI], [SI], [M]$_\tau$ and [C] if and only if it is DQ_τ^ρ for some decreasing family ρ of monetary scalable risk measures.

- If $\mathrm{DQ}_\tau^\rho(X, \dots, X) = 1$, then $\inf\{\beta \in I : \rho_\beta(X) \le \tau(X)\} = 1$
- To get $\rho_1 = \tau$, one needs this and some continuity (unclear!)

Connecting DQ and DR

Proposition 1

For a given $\tau : \mathcal{X} \to \mathbb{R}_+$, we have $\mathrm{DQ}^{\rho}_{\alpha} = \mathrm{DR}^{\tau}$ where $\rho = (\tau/\alpha)_{\alpha \in (0,\infty)}$. The same holds if $\rho = (b\mathbb{E} + c\tau/\alpha)_{\alpha \in (0,\infty)}$ for some $b \in \mathbb{R}$ and $c > 0$ and $\mathcal{X} = L^1$.

- $\mathrm{DR}^{\mathrm{var}}$ and $\mathrm{DR}^{\mathrm{SD}}$ are special cases of DQ
- If τ satisfies $[\mathrm{CA}]_0$, then $\rho_\alpha = b\mathbb{E} + c\tau/\alpha$ satisfies $[\mathrm{CA}]_b$
- $b\mathbb{E} + c\tau/\alpha$ includes mean-standard deviation, mean-variance, and mean-Gini (Denneberg'90)

Connecting DQ and DR

- $[\pm]$: $\tau \geq 0$ or $\tau \leq 0$

Proposition 2

Fix $n \geq 3$. Assume that $\tau : L^p \to \mathbb{R}$ is law invariant and continuous, and DR^τ is not degenerate. Then, DR^τ satisfies $[+]$, $[\mathrm{LI}]$ and $[\mathrm{SI}]$ if and only if τ satisfies $[\pm]$, $[\mathrm{CA}]_0$ and $[\mathrm{PH}]$.

- If DR^τ satisfies $[+]$, [LI] and [SI]
  - τ must be a variability measure (Furman/W./Zitikis'17) or deviation measure (Rockafellar/Uraysev/Zabarankin'06)
  - τ cannot be monotone like a monetary risk measure

All non-degenerate DR satisfying $[+]$, [LI] and [SI] are in fact DQ

Progress

DQ based on VaR and ES

Theorem 4

For $\alpha \in (0,1)$ *and* $\mathbf{X} \in \mathcal{X}^n$, *by writing* $s_\alpha = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$, $t_\alpha = \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)$ *and* $S = \sum_{i=1}^n X_i$, *we have*

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}\left(S > s_\alpha\right) \quad \textit{and} \quad \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}\left(\mathrm{ES}_U(S) > t_\alpha\right),$$

where $U \sim \mathrm{U}[0,1]$. *Moreover, if* $\mathbb{P}(S > t_\alpha) > 0$, *then*

$$\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = \frac{1}{\alpha} \min_{r \in (0,\infty)} \mathbb{E}\left[\left(r\left(S - t_\alpha\right) + 1\right)_+\right],$$

and otherwise $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = 0$.

- Inverting the ES curve: bPOE of Mafusalov/Uryasev'18

DQ based on VaR and ES

Proof of the last statement.

$$
\begin{aligned}
\mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X}) &= \frac{1}{\alpha}\inf\{\beta\in(0,1): \mathrm{ES}_\beta(S)-t_\alpha\le 0\}\\
&= \frac{1}{\alpha}\inf\{\beta\in(0,1): \mathrm{ES}_\beta(S-t_\alpha)\le 0\}\\
(*) &= \frac{1}{\alpha}\inf\left\{\beta\in(0,1): \min_{t\in\mathbb{R}}\left\{t+\frac{1}{\beta}\mathbb{E}\left[(S-t_\alpha-t)_+\right]\right\}\le 0\right\}\\
&= \frac{1}{\alpha}\inf\left\{\beta\in(0,1): \exists t\in\mathbb{R} \text{ s.t. } \frac{1}{\beta}\mathbb{E}\left[(S-t_\alpha-t)_+\right]\le -t\right\}\\
&= \frac{1}{\alpha}\inf\left\{\beta\in(0,1): \exists r>0 \text{ s.t. } \mathbb{E}\left[(r(S-t_\alpha)+1)_+\right]\le \beta\right\}\\
&= \frac{1}{\alpha}\inf_{r>0}\mathbb{E}\left[(r(S-t_\alpha)+1)_+\right]
\end{aligned}
$$

(*): Rockafellar/Uryasev'02

Comparing DQ and DR on VaR

Comparing

$$\mathrm{DQ}_{\alpha}^{\mathrm{VaR}}(\mathbf{X}) = \frac{\mathbb{P}(S > s_{\alpha})}{\alpha} \quad \text{and} \quad \mathrm{DR}^{\mathrm{VaR}_{\alpha}}(\mathbf{X}) = \frac{\mathrm{VaR}_{\alpha}(S)}{s_{\alpha}}$$

Duality:

- DQ measures the "probability improvement"
- DR measures the "quantile improvement"

Generally, none of DQ and DR based on VaR or ES is convex

Dependence and portfolio risks

For a random variable X and $\alpha \in (0,1)$

(i) A tail event of X is an event $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$ such that $X(\omega) \geq X(\omega')$ holds for a.s. all $\omega \in A$ and $\omega' \in A^c$

(ii) A random vector $(X_1, ..., X_n)$ is α-concentrated if its component share a common tail event of probability α (W./Zitikis'21)

For $\mathbf{X} \in \mathcal{X}^n$ and $\alpha \in (0, 1/n)$, an α-CE model satisfies

- $\mathbb{P}(X_i > \mathrm{VaR}_\alpha(X_i)) = \alpha$
- $\mathbb{P}(X_i \geq \mathrm{VaR}_\alpha(X_i)) \geq n\alpha$
- $(X_1, \dots, X_n)$ are $(n\alpha)$-concentrated
- $\{X_i > \mathrm{VaR}_\alpha(X_i)\}$, $i = 1, \dots, n$, are mutually exclusive

Dependence and portfolio risks

Theorem 5

Let $\alpha \in (0,1)$ and $n \geq 2$ satisfy $n \leq 1/\alpha$.

(i) $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ *has a range* $[0, n]$ *and* $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ *has a range* $[0, 1]$.

(ii) *If* $\sum_{i=1}^n X_i$ *is a constant, then* $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = 0$.

(iii) *For* ρ *being* VaR *or* ES, *if* $\mathbf{X}$ *is* α*-concentrated, then* $\mathrm{DQ}_\alpha^{\rho}(\mathbf{X}) \leq 1$. *If, in addition,* ρ *is continuous and non-flat from the left at* $(\alpha, \sum_{i=1}^n X_i)$, *then* $\mathrm{DQ}_\alpha^{\rho}(\mathbf{X}) = 1$.

(iv) *If* $\mathbf{X}$ *has an* α*-CE model, then* $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = n$ *and* $\mathrm{DQ}_{n\alpha}^{\mathrm{ES}}(\mathbf{X}) = 1$.

Asymptotic behaviour of DQ

Proposition 3

Suppose that $X_1, \dots, X_n$ are iid random variables. If $X_1 \in \mathrm{RV}_\gamma$ has positive density over its support, then $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to n^{1-\gamma}$ as $\alpha \downarrow 0$.

- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \approx n$ for ultra heavy-tailed iid model ($\gamma \downarrow 0$)
- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = n$ for α-CE model which is complicated and involves both positive and negative dependence

Progress

Optimal portfolio diversification

- Losses from n assets: $\mathbf{X} \in \mathcal{X}^n$
- Portfolio weight vector: $\mathbf{w}$ such that

$$\mathbf{w} = (w_1, \dots, w_n) \in \Delta_n := \{\mathbf{x} \in [0,1]^n : x_1 + \dots + x_n = 1\}$$

- Portfolio loss vector: $\mathbf{w} \odot \mathbf{X} = (w_1 X_1, \dots, w_n X_n)$
- The total loss of a portfolio: $\mathbf{w}^\top \mathbf{X}$

Optimal one-period diversification problem

$$\min_{\mathbf{w} \in \Delta_n} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X}) \quad \text{and} \quad \min_{\mathbf{w} \in \Delta_n} \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w} \odot \mathbf{X}) \qquad \text{(OD)}$$

Optimal portfolio diversification

Proposition 4

Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a class of convex risk measures satisfying $[\mathrm{PH}]_1$ and decreasing in α. For every $\mathbf{X} \in \mathcal{X}^n$ and $\alpha \in I$, $\mathbf{w} \mapsto \mathrm{DQ}^\rho_\alpha(\mathbf{w} \odot \mathbf{X})$ is quasi-convex.

Remark. For any diversification index D,

- Convexity or quasi-convexity of $\mathbf{X} \mapsto D(\mathbf{X})$ is not desirable
  - For well-diversified (X, Y) and $Z = (X + Y)/2$, we want $D(Z, Z)$ to be larger than both $D(X, Y)$ and $D(Y, X)$
- Convexity of $\mathbf{w} \mapsto D(\mathbf{w} \odot \mathbf{X})$ is not desirable

Optimal portfolio diversification

- Write $\mathbf{x}_\alpha^\rho = (\rho_\alpha(X_1), \dots, \rho_\alpha(X_n))$

Proposition 5

For $\rho = \mathrm{VaR}$, *if each component of* $\mathbf{X}$ *is non-constant, then* (OD) *is solved by*

$$\min_{\mathbf{w}\in\Delta_n} \mathbb{P}\left(\mathbf{w}^\top\left(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{VaR}}\right) > 0\right).$$

For $\rho = \mathrm{ES}$, *if* $\mathbb{P}(\mathbf{w}^\top(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{ES}}) = 0) = 0$ *for all* $\mathbf{w} \in \Delta_n$, *then* (OD) *is solved by*

$$\min_{\mathbf{v}\in\mathbb{R}_+^n} \mathbb{E}\left[\left(\mathbf{v}^\top\left(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{ES}}\right) + 1\right)_+\right],$$

and the optimal $\mathbf{w}$ *is given by* $\mathbf{v}/\|\mathbf{v}\|_1$.

Portfolio optimization of DQ for a data sample

$\min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X})$ for data sample $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$

$\Longrightarrow$ convex programming

$$\text{minimize} \quad \sum_{j=1}^{N}\left(\mathbf{v}^\top\left(\mathbf{X}^{(j)}-\widehat{\mathbf{x}}_\alpha^{\mathrm{ES}}\right)+1\right)_+ \quad \text{over } \mathbf{v}\in\mathbb{R}_+,$$

where $\widehat{\mathbf{x}}_\alpha^{\mathrm{ES}}$ is the empirical version of $\mathbf{x}_\alpha^{\mathrm{ES}}$ based on the sample

- Practically use $\|\mathbf{v}\|_1 \le M$ for a large M, e.g., $M = 100$
- Apply a tie-breaking rule if needed

Portfolio optimization of DQ for a data sample

$\min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{w}\odot\mathbf{X})$ for data sample $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$

$\Longrightarrow$ linear integer programming

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{N} z_j \quad \left(=\textstyle\sum_{j=1}^{N} \mathbb{1}_{\{\mathbf{w}^\top\mathbf{y}^{(j)}>0\}}\right) && \text{(LIP)} \\ \text{subject to} \quad & \mathbf{w}^\top\mathbf{y}^{(j)} - Mz_j \le 0, \quad \textstyle\sum_{j=1}^{n} w_j = 1 \\ & z_j \in \{0,1\}, \quad w_j \ge 0 \ \text{ for all } j\in\{1,\dots,N\}, \end{aligned}$$

where

- $\mathbf{y}^{(j)} = \mathbf{X}^{(j)} - \widehat{\mathbf{x}}^{\mathrm{VaR}}_{\alpha}$
- $\widehat{\mathbf{x}}^{\mathrm{VaR}}_{\alpha}$ is the empirical version of $\mathbf{x}^{\mathrm{VaR}}_{\alpha}$ based on the sample
- $M>0$: $z_j = 1 \Longrightarrow \mathbf{w}^\top\mathbf{y}^{(j)} - Mz_j \le 0$ (the Big M method)

Portfolio optimization of DQ for a data sample

Tie-breaking

- the objective function of (LIP) takes integer values
- let m^* be the optimal value of (LIP)
- pick the closest one (in L^1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$) to a given benchmark $\mathbf{w}_0$ among tied optimizers

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{w} - \mathbf{w}_0\|_1 \\ \text{subject to} \quad & \sum_{j=1}^{N} \mathbb{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)} > 0\}} \leq m^* \\ & \mathbf{w} \in \Delta_n \end{aligned}$$

Outline

Elliptical models

- Elliptical distributions are popular in QRM
- Two examples: normal and t distributions
- Fundamental theorem of QRM: for elliptical models, any PH risk measures are "equivalent" (Embrechts'19 keynote at IME)
- Explicit formulas for $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ are available
- Asymptotic results for $n \to \infty$ and $\alpha \downarrow 0$ are available

Numerical examples: normal and t-distributions

Consider two dispersion matrices, parametrized by $r \in [0,1]$ and $n \in \mathbb{N}$,

- Equicorrelation

$$\Sigma_1 = (\sigma_{ij})_{n \times n}, \qquad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r \text{ for } i \neq j,$$

- Autoregressive AR(1)

$$\Sigma_2 = (\sigma_{ij})_{n \times n}, \qquad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r^{|j-i|} \text{ for } i \neq j.$$

Let $\mathbf{X}_i \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_i)$ and $\mathbf{Y}_i \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma_i)$, $i = 1, 2$

DQ for varying α

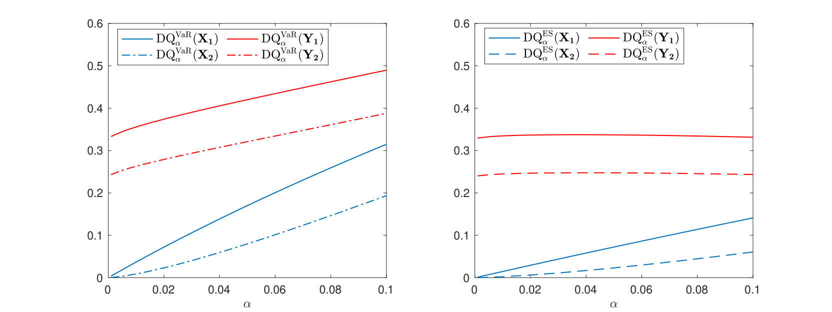


Figure: DQ based on VaR and ES for $\alpha \in (0, 0.1)$ with fixed $\nu = 3$, $r = 0.3$ and $n = 4$

DQ for varying correlation coefficient

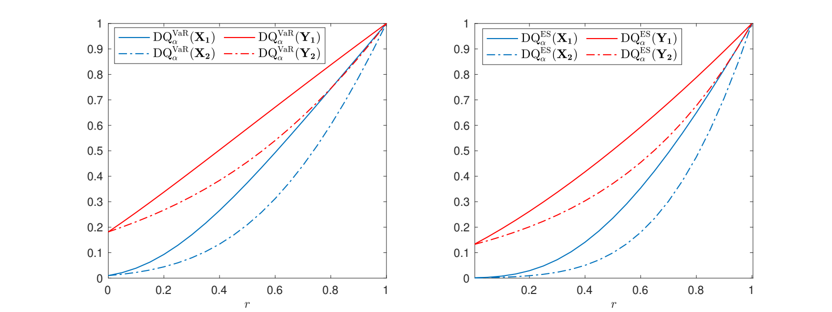


Figure: DQ based on VaR and ES for $r \in [0,1]$ with fixed $\alpha = 0.05$, $\nu = 3$, and $n = 4$

DQ for t-models with varying tail parameter

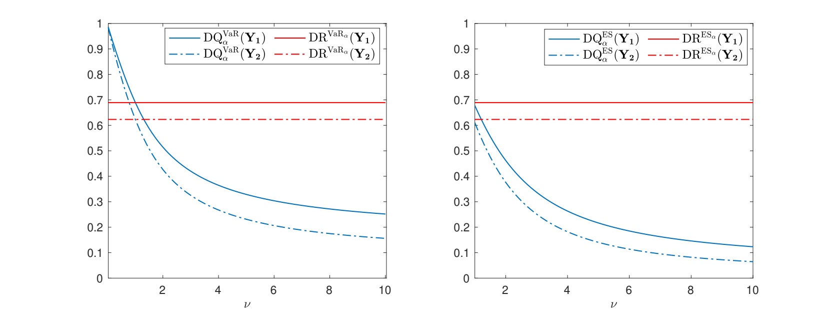


Figure: DQ based on VaR for $\nu \in (0, 10]$ and ES for $\nu \in (1, 10]$ with fixed $\alpha = 0.05$, $r = 0.3$ and $n = 4$

DQ for elliptical models as the dimension n varies

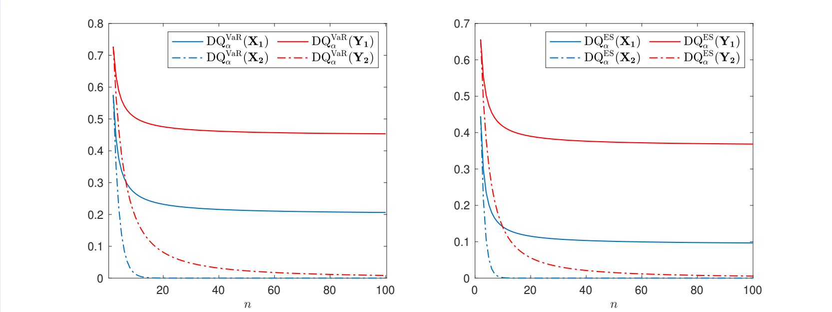


Figure: DQ based on VaR and ES for $n \in [2, 100]$ with fixed $\alpha = 0.05$, $r = 0.5$ and $\nu = 3$

The joint t-model has a common shock

Table: DQ/DR based on VaR, ES and SD, where $\alpha = 0.05$ and $n = 10$

D	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$
$\mathbf{Y}' \sim \mathrm{it}_n(3)$	0.0231	0.0144	0.3568	0.3058	0.3162
$\mathbf{Y} \sim \mathrm{t}(3, \mathbf{0}, I_n)$	0.0502	0.0340	0.3162	0.3162	0.3162
$D(\mathbf{Y}')/D(\mathbf{Y})$	0.4602	0.4235	1.1284	0.9671	1
$\mathbf{Y}' \sim \mathrm{it}_n(4)$	0.0049	0.0024	0.3384	0.2977	0.3162
$\mathbf{Y} \sim \mathrm{t}(4, \mathbf{0}, I_n)$	0.0252	0.0138	0.3162	0.3162	0.3162
$D(\mathbf{Y}')/D(\mathbf{Y})$	0.1944	0.1739	1.0702	0.9415	1
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	$\sim 10^{-6}$	$\sim 10^{-9}$	0.3162	0.3162	0.3162

- DQ: iid normal $<$ iid t $<$ joint t
- DR: iid normal $=$ joint t $\approx$ iid t
- DQ captures tail heaviness/common shock which DR ignores

DQ for t-models with varying tail parameter

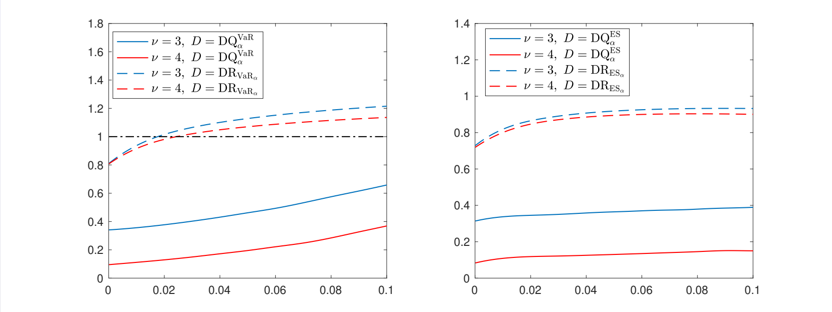


Figure: $D(\mathbf{Y}')/D(\mathbf{Y})$ based on VaR and ES for $\alpha \in (0, 0.1]$ with fixed $n = 10$

Optimization for the elliptical models

- Optimal diversification for DQ ($\boldsymbol{\sigma}$ is the diagonal of Σ)

$$\operatorname*{arg\,min}_{\mathbf{w}\in\Delta_n} \mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{w}\odot\mathbf{X}) = \operatorname*{arg\,min}_{\mathbf{w}\in\Delta_n} \frac{\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}}{\mathbf{w}^\top\boldsymbol{\sigma}}$$

- Optimal diversification for DR

$$\operatorname*{arg\,min}_{\mathbf{w}\in\Delta_n} \mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{w}\odot\mathbf{X}) = \operatorname*{arg\,min}_{\mathbf{w}\in\Delta_n} \frac{\mathbf{w}^\top\boldsymbol{\mu} + y_\alpha\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}}{\mathbf{w}^\top\boldsymbol{\mu} + y_\alpha\mathbf{w}^\top\boldsymbol{\sigma}}$$

  where $y_\alpha = \mathrm{VaR}_\alpha(Y)$ and $Y \sim \mathrm{E}_1(0,1,\phi)$

- The two have the same optimizers if $\boldsymbol{\mu} = \mathbf{0}$ and $y_\alpha \neq 0$
- In case $\Sigma = I_n$: $\mathbf{w}^* = (\frac{1}{n}, \dots, \frac{1}{n})$

Outline

DQ for different portfolios

Data: daily losses from S&P 500 constituents

- Period: January 3, 2012 to December 31, 2021
- 2518 daily losses; moving window of 500 days

Portfolios with stock compositions:

(A) 2 largest stocks from each of 10 different sectors of S&P 500

(B) 1 largest stock from each of 5 different sectors of S&P 500

- XOM (ENR), AAPL (IT), BRK/B (FINL), WMT (CONS), GE (INDU)

(C) 5 largest stocks from the Information Technology (IT) sector

(D) 5 largest stocks from the Financials (FINL) sector

DQ for different portfolios

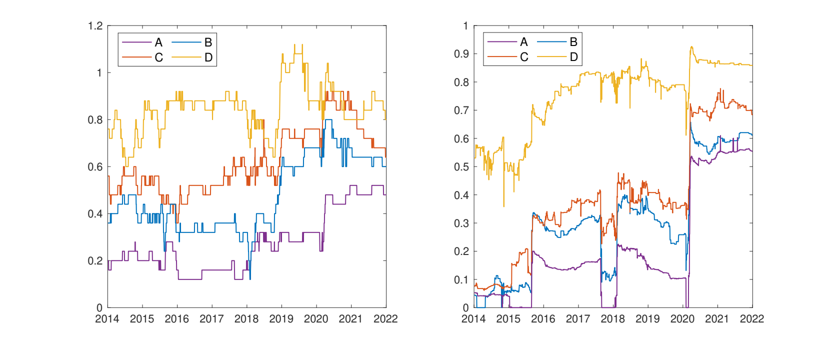


Figure: DQ based on VaR (left) and ES (right) with $\alpha = 0.05$

- Observation: A (20) $<$ B (5) $<$ C (5 IT) $<$ D (5 FINL)
- Large jump for DQ based on ES at the COVID outbreak
- DQ based on VaR can be larger than 1

Optimal diversified portfolios

Portfolios (monthly rebalancing) with 4 largest stocks from each of the 10 sectors of S&P 500 in 2012 (40 in total)

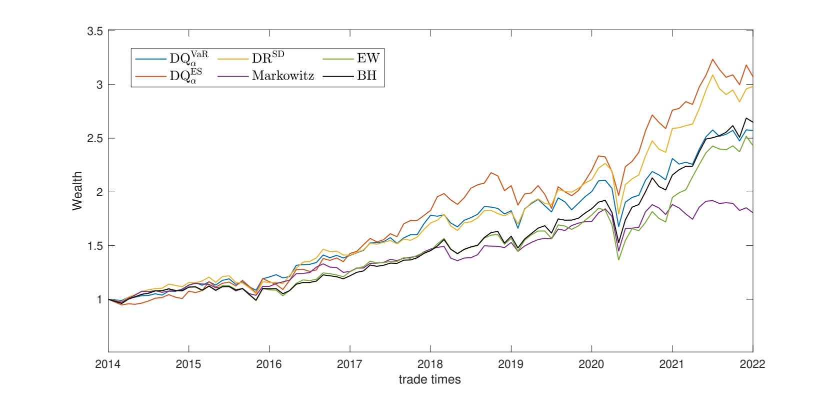


Figure: Wealth processes

Optimal diversified portfolios

%	$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$	$\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	12.562	14.695	14.364	7.929	11.906	12.883
AV	14.643	15.818	14.994	12.976	15.918	14.343
SR	66.397	74.942	76.854	39.222	56.955	70.023

Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021

- Risk-free rate: 2.84% (= 10-y US treasury yield, Jan 2014)
- $\alpha = 0.1$
- EW = equally weighted; BH = buy and hold
- The target AR for the Markowitz portfolio is set to 10%

Optimal diversified portfolios

Portfolios (monthly rebalancing) with 4 largest stocks from each of the 10 sectors of S&P 500 in 2002 (40 in total)

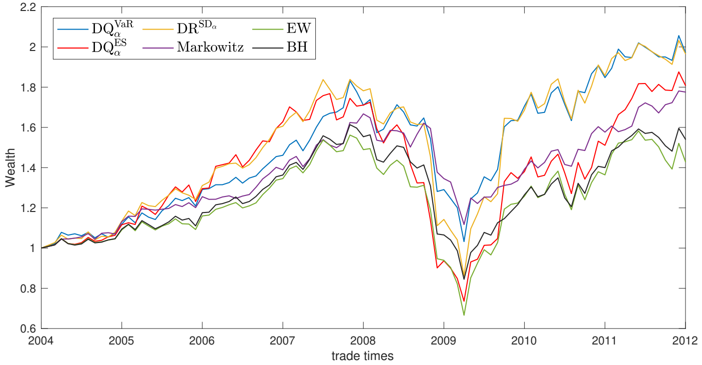


Figure: Wealth processes

Optimal diversified portfolios

%	$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$	$\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	9.456	8.129	9.103	7.980	5.300	6.235
AV	16.653	21.452	20.915	11.976	20.154	15.530
SR	30.478	17.474	22.582	30.064	4.566	11.944

Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2004 to Dec 2011

- Risk-free rate: 4.38% (= 10-y US treasury yield, Jan 2004)
- $\alpha = 0.1$
- EW = equally weighted; BH = buy and hold
- The target AR for the Markowitz portfolio is set to 5%

Outline

Summary

Features

- DQ satisfies [+], [SI] and [LI]
- DQ is intuitive and interpretable with respect to dependence
- DQ can be applied to a wide range of risk measures
  - e.g., VaR, ES, expectiles, entropic risk measures, ...
- DQ includes DR based on variance or standard deviation
- DQ is able to capture heavy tails and common shocks

DQ based on VaR/ES

- DQ has simple formulas and convenient properties
- Portfolio optimization of DQ is efficient and easier than DR
- DQ applied to real data is consistent with usual perception

Some potential limitations of DQ

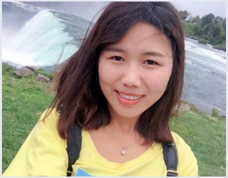
Potential limitations

- DQ requires a class of risk measures
  - all commonly used risk measures are parametric families
  - for a given risk measure, taking its positive part yields a DQ
- Computing DQ requires inverting $\beta \mapsto \rho_\beta(\sum_{i=1}^n X_i)$
  - easy for VaR/ES
  - possibly complicated for other risk measures
- DQ on VaR/ES takes discrete values for empirical data
  - suffers from data scarcity (similar to tail risk measures)

Thank you

Thank you for your kind attention

Based on joint work with



Xia Han
(Nankai)

Liyuan Lin
(Waterloo)

- SSRN: https://ssrn.com/abstract=4149069
- Working papers series: Risk management with risk measures
  http://sas.uwaterloo.ca/~wang/pages/WPS5.html

Some literature on measuring diversification

- Markowitz'52 JF: Mean-variance theory
- Diversification ratio
  - Tasche'07; Choueifaty/Coignard'08 JPM; Bürgi/Dacorogna/Iles'08; Embrechts/Wang/W.'15 FS
- Number of unique investments or naive diversification
  - Rudin/Morgan'06 JPM; DeMiguel/Garlappi/Uppal'09 RFS; Pflug/Pichler/Wozabal'12 JBF
- Diversification benefit in multivariate regular variation models
  - Mainik/Rüchendorf'10 FS; Mainik/Embrechts'13 AAS
- Koumou/Dionne'19: Axioms for correlation diversification measures

Some recent work on VaR and ES

- Axiomatic characterizations
  - VaR: Kou/Peng' 16 OR; He/Peng' 18 OR; Liu/W.'21 MOR
  - ES: W./Zitikis'21 MS; Embrechts/Mao/Wang/W.'21 MF
- Risk sharing
  - Embrechts/Liu/W.'18 OR; Embrechts/Liu/Mao/W.'20 MP
- Robustness in optimization
  - Emberchts/Schied/W.'22 OR
- Calibrating levels between VaR and ES
  - Li/W.'22 JE
- Forecasting and backtesting
  - Fissler/Ziegel'16 AOS; Nolde/Ziegel'17 AOAS; Du/Escanciano'17 MS

Consistency across dimensions

Allow D to be defined on $\bigcup_{n\in\mathbb{N}} \mathcal{X}^n$

[RI] Riskless invariance: $D(\mathbf{X}, c) = D(\mathbf{X})$ for all $n \in \mathbb{N}$, $\mathbf{X} \in \mathcal{X}^n$ and $c \in \mathbb{R}$

- adding a risk-free asset to the portfolio does not affect D

[RC] Replication consistency: $D(\mathbf{X}, \mathbf{X}) = D(\mathbf{X})$ for all $n \in \mathbb{N}$ and $\mathbf{X} \in \mathcal{X}^n$

- replicating the same portfolio composition does not affect D

Consistency across dimensions

Proposition 6

Let $\rho = (\rho_\alpha)_{\alpha \in I}$ *be a class of risk measures decreasing in* α *and* $\tau : L^p \to \mathbb{R}$ *be a continuous and law-invariant risk measure.*

(i) *If* ρ *satisfies* $[\mathrm{CA}]$ *and* $\rho_\alpha(0) = 0$ *for* $\alpha \in I$*, then* DQ^ρ_α *satisfies* $[\mathrm{RI}]$ *for* $\alpha \in I$*.*

(ii) *Suppose that* DR^τ *is not degenerate when the input dimension is at least 2. Then* DR^τ *satisfies* $[\mathrm{RI}]$ *and* $[+]$ *if and only if* τ *satisfies* $[\mathrm{CA}]_0$*,* $[\pm]$ *and* $\tau(0) = 0$*.*

(iii) *If* ρ *satisfies* $[\mathrm{PH}]_1$*, then* DQ^ρ_α *satisfies* $[\mathrm{RC}]$ *for* $\alpha \in I$*.*

(iv) *If* τ *satisfies* $[\mathrm{PH}]_1$*, then* DR^τ *satisfies* $[\mathrm{RC}]$*.*

Elliptical models

- A random vector $\mathbf{X}$ is elliptically distributed if it has a characteristic function

$$\psi(\mathbf{t}) = \mathbb{E}\left[\exp\left(\mathtt{i}\mathbf{t}^\top \mathbf{X}\right)\right] = \exp\left(\mathtt{i}\mathbf{t}^\top \boldsymbol{\mu}\right)\phi\left(\mathbf{t}^\top \Sigma \mathbf{t}\right),$$

  for some $\boldsymbol{\mu} \in \mathbb{R}^n$, positive semi-definite matrix $\Sigma \in \mathbb{R}^{n\times n}$, and $\phi : \mathbb{R}_+ \to \mathbb{R}$ (the characteristic generator)
- This distribution is denoted by by $\mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$
- Write $\Sigma = (\sigma_{ij})_{n\times n}$, $\sigma_i^2 = \sigma_{ii}$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, and

$$k_\Sigma = \frac{\sum_{i=1}^n \sigma_i}{\left(\sum_{i,j}^n \sigma_{ij}\right)^{1/2}} \in [1, \infty)$$

DQ for elliptical models

Proposition 7

Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$. *We have*

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{1 - F(k_\Sigma \mathrm{VaR}_\alpha(Y))}{\alpha}; \ \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = \frac{1 - \widetilde{F}(k_\Sigma \mathrm{ES}_\alpha(Y))}{\alpha},$$

for $\alpha \in (0,1)$, *where* $Y \sim \mathrm{E}_1(0,1,\phi)$ *with distribution function* F, *and* $\widetilde{F}$ *is the superquantile transform of* F. *Moreover,*

(i) $\alpha \mapsto \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ *takes value in* $[0,1]$ *on* $(0,1/2]$ *and it takes value in* $[1,2]$ *on* $(1/2,1)$;

(ii) $k_\Sigma \mapsto \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ *is decreasing for* $\alpha \in (0,1/2]$ *and increasing for* $\alpha \in (1/2,1)$;

(iii) $k_\Sigma \mapsto \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$ *is decreasing for* $\alpha \in (0,1)$.

Asymptotic behaviour of DQ

Proposition 8

Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$.

(i) *Let* $Y \sim \mathrm{E}_1(0, 1, \phi)$ *and* f *be the density function of* Y. *We have*

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{x \to \infty} k_\Sigma \frac{f(k_\Sigma x)}{f(x)}$$

if $\mathrm{VaR}_0(Y) = \infty$ *and the limit exists, and* $\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = 0$ *if* $\mathrm{VaR}_0(Y) < \infty$.

(ii) *Let* $\mathrm{AC}_\Sigma = 1/k_\Sigma^2$. *If* $\lim_{n \to \infty} \mathrm{AC}_\Sigma = 0$, *then*

$$\lim_{n \to \infty} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{n \to \infty} \mathrm{DQ}_\beta^{\mathrm{ES}}(\mathbf{X}) = 0$$

for $\alpha \in (0, 1/2)$ *and* $\beta \in (0, 1)$.

Cross-comparison between DQ based on VaR and ES

Associating VaR and ES levels by PELVE (Li/W.'22)

$$\mathrm{ES}_{c\alpha}(X) = \mathrm{VaR}_{\alpha}(X)$$

Table: Values of DQ based on VaR at level $\alpha = 0.01$ and ES at level $c\alpha$, where $n = 4$ and $r = 0.3$

	c	$c\alpha$	$\mathrm{DQ}_{\alpha}^{\mathrm{VaR}}$	$\mathrm{DQ}_{c\alpha}^{\mathrm{ES}}$
$\mathbf{X}_1 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_1)$	2.58	0.0258	0.0369	0.0377
$\mathbf{X}_2 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_2)$	2.58	0.0258	0.0024	0.0025
$\mathbf{Y}_1 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_1)$	3.31	0.0331	0.3558	0.3373
$\mathbf{Y}_2 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_2)$	3.31	0.0331	0.2094	0.1961

Numerical example

Assume that $\mathbf{X} \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma)$ where $\nu = 3$ and the dispersion matrix is given by

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}$$

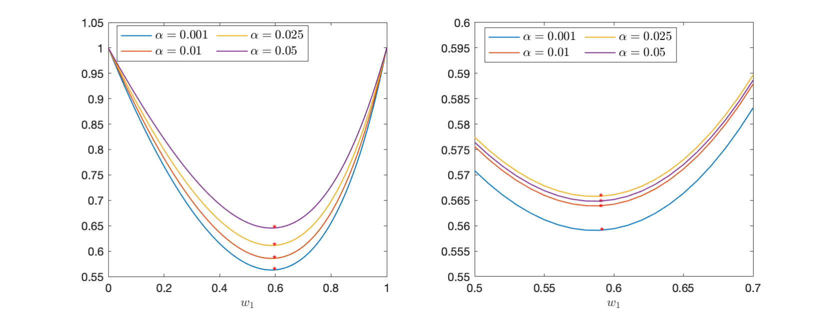


Figure: Values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X})$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w} \odot \mathbf{X})$ for $w_1 \in [0, 1]$

MRV model

Definition 2

A random vector $\mathbf{X} \in \mathcal{X}^n$ is multivariate regularly varying (MRV) with some $\gamma > 0$ if there exists a random vector $\mathbf{\Theta}$ with values in the unit sphere $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}$ such that for any $t > 0$ and any Borel set $S \subseteq \mathbb{S}^{n-1}$ with $P(\mathbf{\Theta} \in \partial S) = 0$,

$$\lim_{x \to \infty} \frac{P(\|\mathbf{X}\| > tx,\ \mathbf{X}/\|\mathbf{X}\| \in S)}{P(\|\mathbf{X}\| > x)} = t^{-\gamma} P(\mathbf{\Theta} \in S),$$

where $\|\cdot\|$ is any fixed norm.

We will use the shorthand notation $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ for MRV models with common tail index γ and spectral measure Ψ

Asymptotic behaviour of DQ

Proposition 9

Suppose that $X_1, \dots, X_n$ are iid random variables. If $X_1 \in \mathrm{RV}_\gamma$ has positive density over its support, then $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to n^{1-\gamma}$ as $\alpha \downarrow 0$.

- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \approx n$ for ultra heavy-tailed iid model ($\gamma \downarrow 0$)
- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = n$ for α-CE model which is complicated and involves both positive and negative dependence

Optimization for the MRV models

Proposition 10

Suppose that $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ *and* $\mathbf{X}$ *has positive joint density on the support of* $\mathbf{X}$. *Then we have*

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X}) = f(\mathbf{w}),$$

where $f(\mathbf{w}) = \eta_{\mathbf{w}} \left(\sum_{i=1}^n w_i \eta_{\mathbf{e}_i}^{1/\gamma} \right)^{-\gamma}$ *and* $\eta_{\mathbf{x}} = \int_{\Delta_n} (\mathbf{x}^\top \mathbf{s})^\gamma \, \Psi(\mathrm{d}\mathbf{s})$ *for* $\mathbf{x} \in \mathbb{R}^n$.

- Approximately optimizing $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ by minimizing $f(\mathbf{w})$

Numerical example

Assume that Y_1 and Y_2 are iid following a standard t-distribution with $\nu > 1$ degrees of freedom. A random vector $\mathbf{X} = (X_1, X_2)$ is defined as

$$\mathbf{X} = A\mathbf{Y} \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ r & \sqrt{1-r^2} \end{pmatrix}.$$

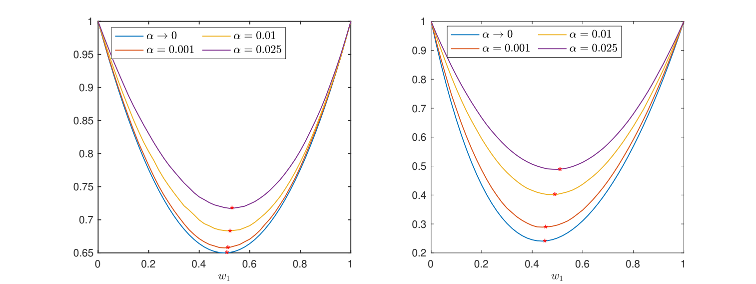


Figure: Values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X})$ with $\nu = 2$ (left) and $\nu = 4$ (right)

Comparison of DQ and DR

- Portfolio: 1 largest stock from each of 5 sectors (2012 market cap)
  - XOM (ENR), AAPL (IT), BRK/B (FINL), WMT (CONS), GE (INDU)
- Period: January 3, 2012 to December 31, 2021
- 2518 daily losses; moving window of 500 days

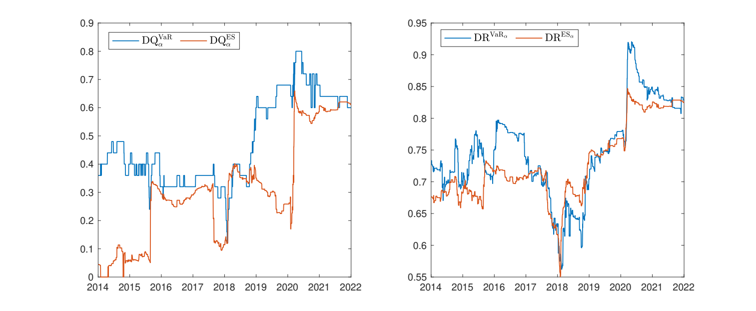


Figure: DQ (left) and DR (right) on VaR/ES with $\alpha = 0.05$

Optimal diversified portfolio

Portfolios (monthly rebalancing) with 2 largest stocks from each of the 10 sectors of S&P 500 in 2012 (20 in total)

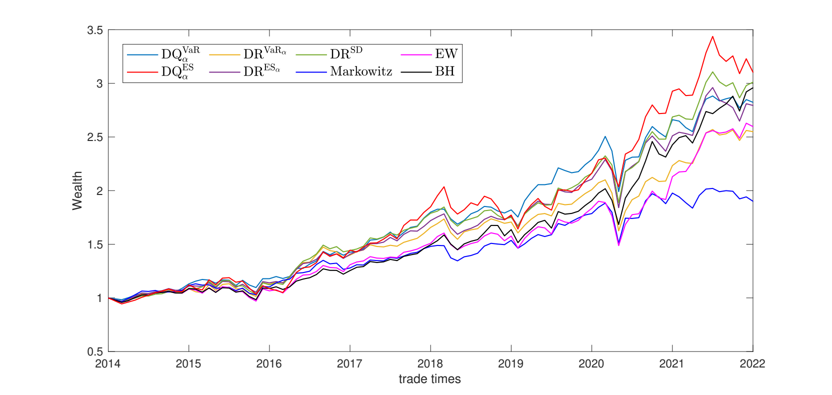


Figure: Wealth processes

Optimal diversified portfolio

%	$\mathrm{DQ}_{\alpha}^{\mathrm{VaR}}$	$\mathrm{DQ}_{\alpha}^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{VaR}_{\alpha}}$	$\mathrm{DR}^{\mathrm{ES}_{\alpha}}$
AR	13.5449	14.4763	12.7657	13.8492
AV	13.4340	15.7689	14.4079	14.5265
SR	79.6853	73.7905	68.8908	75.7867

%	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	14.3663	8.5884	12.7359	14.2236
AV	14.2887	12.7355	14.6834	13.9614
SR	80.6671	45.1371	67.3952	81.5362

Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021