Model Aggregation for Risk Evaluation and Robust Optimization

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Agenda

1. Model uncertainty and robust optimization
2. The model aggregation approach
3. Equivalence in model aggregation
4. Common settings of uncertainty models
5. Empirical applications

Based on joint work with Tiantian Mao (USTC) and Qinyu Wu (USTC)

Distributional uncertainty

Ideal world: stochastic or statistical models are available

- risk evaluation based on specified models $\Leftarrow$ risk measure ρ
- decisions and optimization

Reality: uncertainty is everywhere $\Leftarrow$ uncertainty set $\mathcal{F}$

- statistical uncertainty and data scarcity
- modeling limitations and misspecification
- measurement and mechanistic errors

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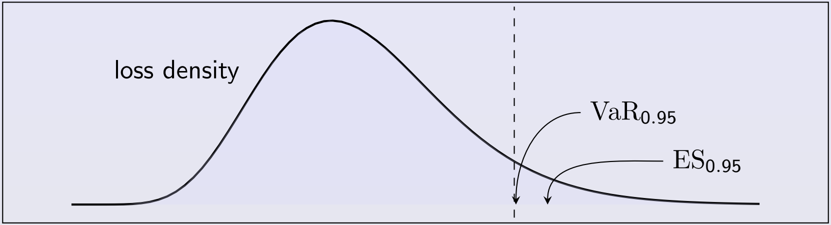
The worst-case risk approach (WR)

$$\rho^{\mathrm{WR}}(\mathcal{F}) := \sup_{F \in \mathcal{F}} \rho(F)$$

- We treat ρ as mappings from either distributions or rvs

VaR and ES



Value-at-Risk (VaR), $\alpha \in (0,1)$

$\mathrm{VaR}_\alpha : \mathcal{M}_0 \to \mathbb{R}$,

$$\mathrm{VaR}_\alpha(F) = F^{-1}(\alpha)$$
$$= \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

(left-quantile)

Expected Shortfall (ES), $\alpha \in (0,1)$

$\mathrm{ES}_\alpha : \mathcal{M}_1 \to \mathbb{R}$,

$$\mathrm{ES}_\alpha(F) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\beta(F)\mathrm{d}\beta$$

(also: TVaR/CVaR/AVaR)

Classical robust optimization

Distributionally robust optimization (DRO)

$$\text{minimize over } \mathbf{a} \in A: \quad \sup_{\mathbf{X} \in \mathcal{X}} \rho(f(\mathbf{a}, \mathbf{X})) = \rho^{\mathrm{WR}}(\mathcal{F}_{\mathbf{a},f})$$

- A: a set of admissible actions
- $\mathcal{X}$: an uncertainty set of possible risk vectors, $\mathbb{R}^d$-valued
- $f : A \times \mathbb{R}^d \to \mathbb{R}$ a loss function
- $\mathcal{F}_{\mathbf{a},f} = \{\text{distribution of } f(\mathbf{a}, \mathbf{X}) : \mathbf{X} \in \mathcal{X}\}$
- Example (portfolio selection): $A \subseteq \mathbb{R}^d$ and $f(\mathbf{a}, \mathbf{x}) = \mathbf{a}^\top \mathbf{x}$
- Various formulations of uncertainty sets in optimization
  - Zhu-Fukushima'09 OR; Natarajan-Pachamanova-Sim'08 MS; Ghaoui-Oks-Oustry'13 OR; Esfahani-Kuhn'18 MP; Gao-Kleywegt'22 MOR; Blanchet-Murthy'19 MOR; Li'18 OR; ...

Our idea

The worst-case risk approach (WR)

$$\text{Uncertainty set } \mathcal{F} + \text{risk measure } \rho \implies \sup_{F \in \mathcal{F}} \rho(F)$$

Our idea

The worst-case risk approach (WR)

Uncertainty set $\mathcal{F}$ + risk measure $\rho \Longrightarrow \sup_{F\in\mathcal{F}} \rho(F)$

The model aggregation approach (MA)

Uncertainty set $\mathcal{F} \Longrightarrow$ A conservative distribution F^* from $\mathcal{F}$

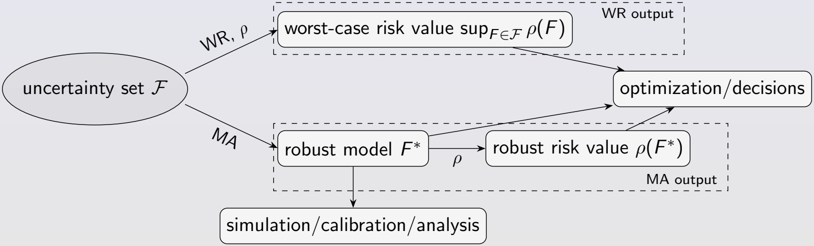
Our idea

The worst-case risk approach (WR)

$$\text{Uncertainty set } \mathcal{F} + \text{risk measure } \rho \Longrightarrow \sup_{F\in\mathcal{F}} \rho(F)$$

The model aggregation approach (MA)

$$\text{Uncertainty set } \mathcal{F} \Longrightarrow \text{A conservative distribution } F^* \text{ from } \mathcal{F}$$



Questions

- How do we define a conservative distribution F^* from the uncertainty set $\mathcal{F}$?
- What are theoretical features of the MA approach over the WR?
- How do the MA and WR approaches compare to each other, what are the implications?
- How is the MA approach implemented in common settings of uncertainty, optimization, and real-data applications?

Progress

Supremum of a set

An ordered set $(\mathcal{M}_1, \preceq)$

- $\mathcal{M}_1$: the set of all finite-mean distributions
- $\preceq$: a partial order on $\mathcal{M}_1$

G dominates $\mathcal{F}$: $F \preceq G$ for all $F \in \mathcal{F}$

Definition 1 (Supremum of a set)

For $\mathcal{F} \subseteq \mathcal{M}_1$, the supremum of $\mathcal{F}$ with respect to $\preceq$, denoted by $\bigvee \mathcal{F} \in \mathcal{M}_1$, is the smallest distribution in $\mathcal{M}_1$ dominating $\mathcal{F}$.

- $F \preceq \bigvee \mathcal{F} \preceq G$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{M}_1$ dominating $\mathcal{F}$.
- If such G exists, we say that $\mathcal{F}$ is bounded from above with respect to $\preceq$.

Stochastic dominance

Most important orders of risk

- First-order stochastic dominance (FSD, usual stochastic order):

$$F \preceq_1 G \iff \int u \mathrm{d}F \le \int u \mathrm{d}G \text{ for all increasing functions } u$$

- Second-order stochastic dominance (SSD, increasing convex order):

$$F \preceq_2 G \iff \int u \mathrm{d}F \le \int u \mathrm{d}G \text{ for all increasing convex } u$$

$$F \preceq_1 G \iff \mathrm{VaR}_\alpha(F) \le \mathrm{VaR}_\alpha(G) \text{ for all } \alpha \iff G \ge F$$

$$F \preceq_2 G \iff \mathrm{ES}_\alpha(F) \le \mathrm{ES}_\alpha(G) \text{ for all } \alpha \iff \pi_F \le \pi_G$$

- $\pi_F(x) = \int_x^\infty \overline{F}(t)\mathrm{d}t = \mathbb{E}_F[(X-x)_+]$
- $F(x) = 1 + (\pi_F(x))'_+$

Conservative distributions with $\preceq_1$ and $\preceq_2$

Proposition 1

(i) *For $\mathcal{F} \subseteq \mathcal{M}_1$ bounded from above with respect to $\preceq_1$,*

$$\bigvee_1 \mathcal{F} = \inf_{F\in\mathcal{F}} F \quad \text{and} \quad \left(\bigvee_1 \mathcal{F}\right)^{-1} = \sup_{F\in\mathcal{F}} F^{-1}.$$

(ii) *For $\mathcal{F} \subseteq \mathcal{M}_1$ bounded from above with respect to $\preceq_2$,*

$$\bigvee_2 \mathcal{F} = 1 + \left(\sup_{F\in\mathcal{F}} \pi_F\right)'_+ \quad \text{and} \quad \pi_{\bigvee_2 \mathcal{F}} = \sup_{F\in\mathcal{F}} \pi_F.$$

Proposition 2

For $i \in \{1,2\}$ and $\mathcal{F} \subseteq \mathcal{M}_1$, $\bigvee_i \mathrm{conv}\mathcal{F} = \bigvee_i \mathcal{F}$, where $\mathrm{conv}\mathcal{F}$ is the convex hull of $\mathcal{F}$. $\Longrightarrow$ *no extra difficulty with non-convexity!*

WR and MA approaches

Define

$$\rho^{\mathrm{WR}}(\mathcal{F}) = \sup_{F\in\mathcal{F}} \rho(F) \quad \text{and} \quad \rho^{\mathrm{MA}}(\mathcal{F}) = \rho\left(\bigvee \mathcal{F}\right) \qquad \text{(omitting } \preceq\text{)}$$

For the uncertainty described by $\mathcal{F}_{\mathbf{a},f}$, two optimization approaches

$$\min_{\mathbf{a}\in A} \ \rho^{\mathrm{WR}}(\mathcal{F}_{\mathbf{a},f}) \quad \text{and} \quad \min_{\mathbf{a}\in A} \ \rho^{\mathrm{MA}}(\mathcal{F}_{\mathbf{a},f})$$

- WR: quite difficult to solve
  - repeatedly computing $\rho(f(\mathbf{a},\mathbf{X}))$ for every $\mathbf{a}$ and every $\mathbf{X}$
  - non-convexity of the uncertainty set causes problem
- MA: more tractable
  - ρ is only computed once
  - non-convexity is not a problem
  - robust model available

MA approach in robust optimization

MA for ES and $\preceq_2$: write $\beta = 1/(1-\alpha)$

$$\mathrm{ES}_\alpha(F) = \min_{x\in\mathbb{R}} \{x + \beta\pi_F(x)\} \quad \text{(Rockafellar-Uyrasev'02 JBF)}$$

$$\mathrm{ES}^{\mathrm{WR}}_\alpha(\mathcal{F}) = \sup_{F\in\mathcal{F}} \mathrm{ES}_\alpha(F) = \sup_{F\in\mathcal{F}}\min_{x\in\mathbb{R}} \{x + \beta\pi_F(x)\}$$

$$\mathrm{ES}^{\mathrm{MA}}_\alpha(\mathcal{F}) = \mathrm{ES}_\alpha\left(\bigvee\nolimits_2 \mathcal{F}\right) = \min_{x\in\mathbb{R}}\sup_{F\in\mathcal{F}} \{x + \beta\pi_F(x)\}$$

$$\min_{\mathbf{a}\in A} \mathrm{ES}^{\mathrm{WR}}_\alpha(\mathcal{F}_{\mathbf{a},f}) = \min_{\mathbf{a}\in A}\sup_{\mathbf{X}\in\mathcal{X}}\min_{x\in\mathbb{R}} \{x + \beta\mathbb{E}[(f(\mathbf{a},\mathbf{X})-x)_+]\}$$

$$\min_{\mathbf{a}\in A} \mathrm{ES}^{\mathrm{MA}}_\alpha(\mathcal{F}_{\mathbf{a},f}) = \min_{\mathbf{a}\in A, x\in\mathbb{R}}\sup_{\mathbf{X}\in\mathcal{X}} \{x + \beta\mathbb{E}[(f(\mathbf{a},\mathbf{X})-x)_+]\}$$

- $\mathrm{ES}^{\mathrm{WR}}_\alpha(\mathcal{F}) \le \mathrm{ES}^{\mathrm{MA}}_\alpha(\mathcal{F})$ always hold
- Equivalence under some conditions of minimax theorems

Progress

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Is the MA approach more prudent?

Fix an ordered set $(\mathcal{M}, \preceq)$

- ρ is consistent with $\preceq$: $F \preceq G \Longrightarrow \rho(F) \leq \rho(G)$
- $\rho^{\mathrm{WR}}(\mathcal{F}) \leq \rho^{\mathrm{MA}}(\mathcal{F}) \Longrightarrow$ MA is more prudent than WR
- Question: when does $\rho^{\mathrm{WR}}(\mathcal{F}) = \rho^{\mathrm{MA}}(\mathcal{F})$ hold?

Is the MA approach more prudent?

Fix an ordered set $(\mathcal{M}, \preceq)$

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- Question: when does $\rho^{\mathrm{WR}}(\mathcal{F}) = \rho^{\mathrm{MA}}(\mathcal{F})$ hold?

Definition 2 ($\preceq$-cEMA)

Let $(\mathcal{M}, \preceq)$ be an ordered set. A mapping $\rho : \mathcal{M} \to \mathbb{R}$ satisfies $\preceq$-cEMA if $\rho\left(\bigvee \mathcal{F}\right) = \sup_{F \in \mathcal{F}} \rho(F)$ holds for all convex sets $\mathcal{F} \subseteq \mathcal{M}$ bounded from above.

Characterization with $\preceq_1$-cEMA

Properties of risk measures

- Translation invariance (TI): $\rho(F_{X+c}) = \rho(F_X) + c$ for all $c \in \mathbb{R}$, rv X
- Positive homogeneity (PH): $\rho(F_{\lambda X}) = \lambda\rho(F_X)$ for all $\lambda > 0$, rv X
- Lower semicontinuity (LS): $\liminf_{n\to\infty} \rho(F_n) \geq \rho(F)$ if $F_n \xrightarrow{\mathrm{d}} F$

Theorem 1

(i) *A mapping $\rho : \mathcal{M}_1 \to \mathbb{R}$ satisfies TI, PH, LS and $\preceq_1$-cEMA if and only if $\rho = \mathrm{VaR}_\alpha$ for some $\alpha \in (0, 1)$.*

(ii) *A mapping $\rho : \mathcal{M}_1 \to \mathbb{R}$ satisfies TI, PH, LS and $\preceq_2$-cEMA if and only if $\rho = \mathrm{ES}_\alpha$ for some $\alpha \in (0, 1)$.*

- Sufficient if cEMA is imposed only for convex sets with two extreme points

Characterization with cEMA

Axiomatic characterizations of VaR (quantile): key axioms

- Chambers'09 MF: ordinal covariance + law invariance
- Kou-Peng'16 OR: elicitability + comonotonic additivity
- He-Peng'18 OR: surplus invariance + law invariance + PH
- Liu-W.'21 MOR: elicitability + tail relevance + PH

Axiomatic characterizations of ES: key axioms

- W.-Zitikis'21 MS: no reward for concentration
- Embrechts-Mao-Wang-W'21 MF: elicitability + Bayes risk
- Han-Wang-W.-Wu'21 wp: TI + concentration aversion

EMA for arbitrary uncertainty sets

- $\preceq$-EMA: $\rho\left(\bigvee \mathcal{F}\right) = \sup_{F\in\mathcal{F}} \rho(F)$ for $\mathcal{F} \subseteq \mathcal{M}$ bounded from above
- $\rho(\delta_0) = 0$, TI, LS and $\preceq_1$-EMA

  $\iff \rho(F) = \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(F) - h(\alpha)\}$ for some increasing h ...

  $\iff$ benchmark-adjusted VaR (Bignozzi-Burzoni-Munari'20 JRI)
- $\rho(\delta_0) = 0$, TI and $\preceq_2$-EMA

  $\iff \rho(F) = \sup_{\alpha\in[0,1)} \{\mathrm{ES}_\alpha(F) - g(\alpha)\}$ for some increasing g ...

  $\iff$ benchmark-adjusted ES (Burzoni-Munari-W.'22 JBF)
- ES does not satisfy $\preceq_2$-EMA

Progress

Moment uncertainty

Mean-variance uncertainty set

$$\mathcal{F}_{\mu,\sigma} = \left\{F \in \mathcal{M}_2 : \mathbb{E}[F] = \mu \text{ and } \mathrm{Var}(F) = \sigma^2\right\}$$

- Let $F^1_{\mu,\sigma} = \bigvee_1 \mathcal{F}_{\mu,\sigma}$ and $F^2_{\mu,\sigma} = \bigvee_2 \mathcal{F}_{\mu,\sigma}$
- Robust distributions are explicit

$$F^1_{\mu,\sigma}(x) = \frac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2}, \quad x \geq \mu$$

$$F^2_{\mu,\sigma}(x) = \frac{1}{2}\left(1 + \frac{x-\mu}{\sqrt{(x-\mu)^2+\sigma^2}}\right), \quad x \in \mathbb{R}$$

- Many risk measures ρ admit explicit formulas for $\rho^{\mathrm{MA}}(\mathcal{F}_{\mu,\sigma})$

MA for robust portfolio optimization

- Mean and covariance uncertainty set

$$\mathcal{F}_{\mathbf{w},\mu,\Sigma} = \{F_{\mathbf{w}^\top \mathbf{X}} : \mathbb{E}[\mathbf{X}] = \mu,\ \mathrm{Cov}(\mathbf{X}) = \Sigma\}$$

- The robust portfolio selection equivalence (Popescu'07 OR)

$$\min_{\mathbf{w}\in\mathcal{W}} \rho\left(\bigvee \mathcal{F}_{\mathbf{w},\mu,\Sigma}\right) = \min_{\mathbf{w}\in\mathcal{W}} \rho\left(\bigvee \mathcal{F}_{\mathbf{w}^\top \mu, \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}}\right)$$

- ρ satisfies TI and PI $\Longrightarrow$ second-order conic program, for $\preceq_i$,

$$\min_{\mathbf{w}\in\mathcal{W}} \rho^{\mathrm{MA}}\left(\mathcal{F}_{\mathbf{w}^\top \boldsymbol{\mu}, \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}}\right) = \min_{\mathbf{w}\in\mathcal{W}} \left\{\mathbf{w}^\top \boldsymbol{\mu} + \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}\ \rho\left(F^i_{0,1}\right)\right\}$$

Univariate Wasserstein uncertainty

- F_0: a reference model
- For $p \geq 1$, the ℓ_p-Wasserstein distance between F and F_0:

$$W_p(F, F_0) = \left(\int_0^1 |F^{-1}(s) - F_0^{-1}(s)|^p \mathrm{d}s \right)^{1/p}$$

- Wasserstein uncertainty set for $\epsilon \geq 0$

$$\mathcal{F}_{p,\epsilon}(F_0) = \{F \in \mathcal{M}_p : W_p(F, F_0) \leq \epsilon\}$$

- Denote by

$$F^1_{p,\epsilon|F_0} = \bigvee_1 \mathcal{F}_{p,\epsilon}(F_0) \quad \text{and} \quad F^2_{p,\epsilon|F_0} = \bigvee_2 \mathcal{F}_{p,\epsilon}(F_0)$$

Conservative distribution for Wasserstein uncertainty

Theorem 2

Suppose that $\epsilon > 0$, $p \geq 1$ *and* $F_0 \in \mathcal{M}_p$.

(a) *The left quantile of* $F^1_{p,\epsilon|F_0}$ *is given by uniquely solving*

$$\left(\int_\alpha^1 \left((F^1_{p,\epsilon|F_0})^{-1}(\alpha) - F_0^{-1}(s)\right)_+^p \mathrm{d}s\right)^{1/p} = \epsilon, \ \ \alpha \in (0,1).$$

(b) *For* $p > 1$, *the left quantile of* $F^2_{p,\epsilon|F_0}$ *is given by*

$$(F^2_{p,\epsilon|F_0})^{-1}(\alpha) = F_0^{-1}(\alpha) + \left(1 - \frac{1}{p}\right)(1-\alpha)^{-1/p}\epsilon, \ \ \alpha \in (0,1).$$

Multivariate Wasserstein uncertainty

- The ℓ_p-Wasserstein distance on $\mathbb{R}^d$, $a, p \geq 1$

$$W^d_{a,p}(F, G) = \inf_{\mathbf{X} \sim F,\ \mathbf{Y} \sim G} \left(\mathbb{E}[\|\mathbf{X} - \mathbf{Y}\|_a^p]\right)^{1/p}$$

- Uncertainty set for the portfolio loss $\mathbf{w}^\top \mathbf{X}$, $\epsilon \geq 0$

$$\mathcal{F}_{\mathbf{w},a,p,\epsilon}(F_0) = \left\{F_{\mathbf{w}^\top \mathbf{Z}} : W^d_{a,p}(F_{\mathbf{Z}}, F_0) \leq \epsilon\right\}, \quad F_0 \in \mathcal{M}_p(\mathbb{R}^d)$$

Theorem 3

For $\epsilon \geq 0$ and $a, p > 1$, $F_{\mathbf{X}} \in \mathcal{M}_p(\mathbb{R}^d)$ and $\mathbf{w} \in \mathbb{R}^d$ such that $\mathbf{w}^\top \mathbf{w} \neq 0$, we have

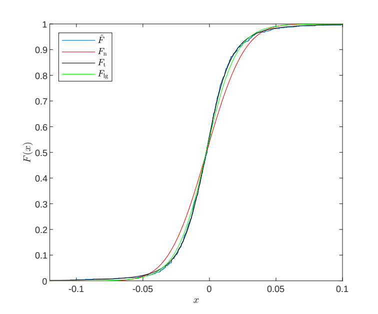
$$\mathcal{F}_{\mathbf{w},a,p,\epsilon}(F_{\mathbf{X}}) = \mathcal{F}_{p,\|\mathbf{w}\|_b \epsilon}(F_{\mathbf{w}^\top \mathbf{X}}),$$

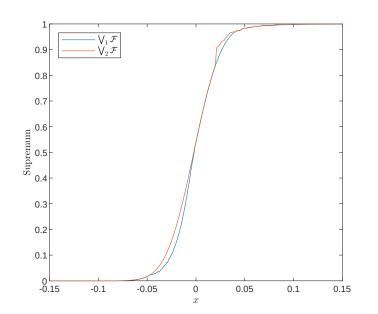
where b satisfies $1/a + 1/b = 1$.

Progress

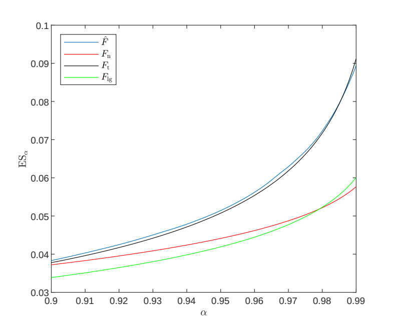
Performance of MA with finite uncertainty set

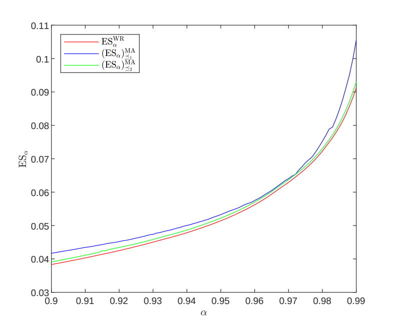
- Daily losses of AAPL from Jan 1, 2019 to Aug 1, 2021
- Fit the data with normal (F_{n}), t (F_{t}), logistic (F_{lg}) models
- $\hat{F}$: the empirical distribution
- Uncertainty set: $\mathcal{F} = \{\hat{F}, F_{\mathrm{n}}, F_{\mathrm{t}}, F_{\mathrm{lg}}\}$





WR and MA for ES





ES for individual models, via WR and via MA

MA approach in robust portfolio selection

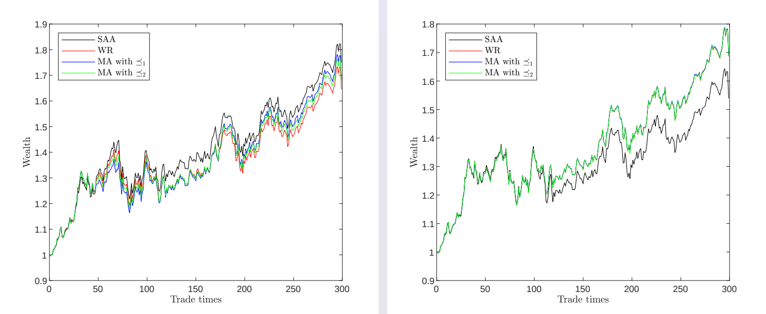
- Daily losses of X_1 (AAPL), X_2 (AMZN), X_3 (EBAY), X_4 (GOOGL) and X_5 (INTC) from Jan 1, 2019 to Aug 1, 2021
- $\mathcal{W} = \{\mathbf{w} \in [0,1]^n : \mathbf{w}^\top \mathbf{1} = 1,\ \mathbf{w}^\top \mathbb{E}[\mathbf{X}] \le -r_0\}$
- Portfolio selection under uncertainty $\mathcal{F}_{\mathbf{w}} = \{F_{\mathbf{w}^\top \mathbf{X}} : F_{\mathbf{X}} \in \mathcal{F}\}$

$$\min_{\mathbf{w}\in\mathcal{W}} \rho^{\mathrm{WR}}(\mathcal{F}_{\mathbf{w}}), \quad \min_{\mathbf{w}\in\mathcal{W}} \rho^{\mathrm{MA}}(\mathcal{F}_{\mathbf{w}}),$$

- $\mathcal{F}$ is modelled by empirical mean-variance or the Wasserstein distance from the fitted t-distribution
- Power distortion risk measure

$$\rho(F) = \int_0^1 k s^{k-1} \mathrm{VaR}_s(F) \mathrm{d}s, \quad k \ge 1$$

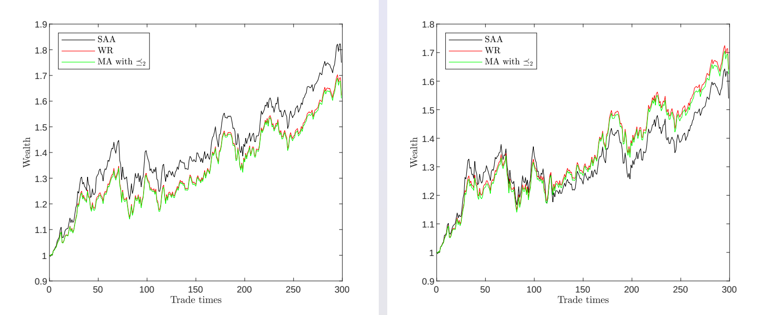
Wealth processes (mean-variance)



Wealth evolution under mean-variance uncertainty ($r_0 = 0.0015$)

Left: $k = 2$; Right: $k = 20$

Wealth processes (Wasserstein with benchmark t-model)



Wealth evolution under Wasserstein uncertainty ($\epsilon = 0.01$, $r_0 = 0.0015$)

Left: $k = 2$; Right: $k = 20$

Concluding remarks

- Both MA and WR are natural to interpret
- MA is motivated by robust distributional models
  - can be used for calibration, analysis, and simulation
  - can be applied without a specified risk measure
  - WR gives the risk value instead of the risk model
- MA robust risk value is easier to compute than WR
  - works well with non-convex $\mathcal{F}$
  - explicit formulas often available
  - handles moment and Wasserstein uncertainty nicely
  - easy to optimize
- MA axiomatically characterizes VaR and ES

Future work

- Using other partial orders, e.g., fractional or multivariate stochastic dominance
  - Müller-Scarsini-Tsetlin-Winkler'17 MS; Huang-Tzeng-Zhao'20 MS
- Using a prior measure on $\mathcal{F}$ for asymmetric treatment of models
- Applying MA to many other settings of uncertainty

Thank you

Thank you for your kind attention

EMA for arbitrary uncertainty sets

- $\preceq$-EMA: $\rho(\bigvee \mathcal{F}) = \sup_{F\in\mathcal{F}} \rho(F)$ for $\mathcal{F} \subseteq \mathcal{M}$ bounded from above
- $\rho(\delta_0) = 0$, TI, LS and $\preceq_1$-EMA $\iff$

$$\rho(F) = \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(F) - h(\alpha)\}$$

for some increasing $h : (0,1) \to [0,\infty]$ with $h(0+) = 0$

  - benchmark-adjusted VaR of Bignozzi-Burzoni-Munari'20 JRI
- $\rho(\delta_0) = 0$, TI and $\preceq_2$-EMA $\iff$

$$\rho(F) = \sup_{\alpha\in[0,1)} \{\mathrm{ES}_\alpha(F) - g(\alpha)\}$$

for some increasing $g : [0,1) \to [0,\infty]$ with $g(0+) = 0$ such that $h : \alpha \mapsto (1-\alpha)g(\alpha)$ is concave on $[0,1)$ with $h(1-) > 0$.

  - benchmark-adjusted ES of Burzoni-Munari-W.'22 JBF

Some common risk measures

The Range Value-at-Risk (RVaR) is defined as

$$\mathrm{RVaR}_{\alpha,\beta}(F) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \mathrm{VaR}_s(F)\mathrm{d}s, \quad 0 \le \alpha < \beta \le 1$$

The power-distorted (PD) risk measure is defined as

$$\mathrm{PD}_k(F) = \int_0^1 ks^{k-1}\mathrm{VaR}_s(F)\mathrm{d}s, \quad k \ge 1$$

The expectile, denoted by ex_α, is defined as the unique solution $t = \mathrm{ex}_\alpha(F) \in \mathbb{R}$ to the following equation,

$$\alpha\mathbb{E}[(X-t)_+] = (1-\alpha)\mathbb{E}[(X-t)_-], \quad X \sim F \in \mathcal{M}_1$$

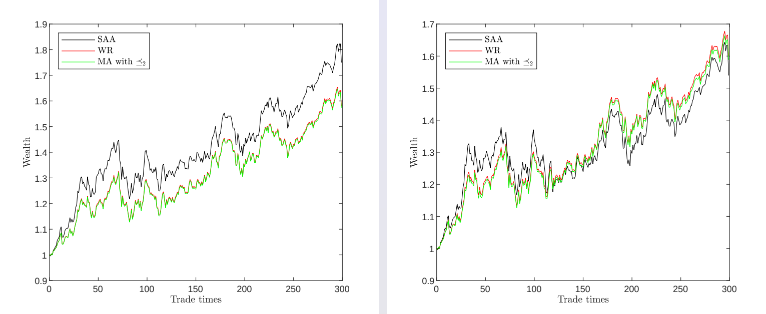
Robust risk measures with MA and WR method

Table: WR and MA under uncertainty induced by $\mathcal{F}_{0,1}$.

ρ	ρ^{WR}	$\rho^{\mathrm{MA}}_{\preceq_1}$	$\rho^{\mathrm{MA}}_{\preceq_2}$
ES_α	$\sqrt{\frac{\alpha}{1-\alpha}}$	$\frac{1}{1-\alpha}\int_\alpha^1 \sqrt{\frac{s}{1-s}}\mathrm{d}s$	$\sqrt{\frac{\alpha}{1-\alpha}}$
$\mathrm{RVaR}_{\alpha,\beta}$	$\sqrt{\frac{\alpha}{1-\alpha}}$	$\frac{1}{\beta-\alpha}\int_\alpha^\beta \sqrt{\frac{s}{1-s}}\mathrm{d}s$	-
VaR_α	$\sqrt{\frac{\alpha}{1-\alpha}}$	$\sqrt{\frac{\alpha}{1-\alpha}}$	-
PD_k	$\frac{k-1}{\sqrt{2k-1}}$	$\frac{\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}$	$\frac{\sqrt{\pi}(k-1)}{2k-1}\frac{\Gamma(k+1/2)}{\Gamma(k)}$
ex_α	$\frac{\alpha-1/2}{\sqrt{\alpha(1-\alpha)}}$	$\mathrm{ex}_\alpha(F^1_{0,1})$	$\frac{\alpha-1/2}{\sqrt{\alpha(1-\alpha)}}$

Γ is the gamma function; $(\mathrm{RVaR}_{\alpha,\beta})^{\mathrm{MA}}_{\preceq_2}$ and $(\mathrm{VaR}_\alpha)^{\mathrm{MA}}_{\preceq_2}$ are not reported because $\mathrm{RVaR}_{\alpha,\beta}$ and VaR_α are not $\preceq_2$-consistent; $\mathrm{ex}_\alpha(\mathcal{F}^1_{0,1})$ can be numerically computed but it does not admit an explicit formula

Wealth processes (Wasserstein with normal benchmark)



Wealth evolution under Wasserstein uncertainty ($\epsilon = 0.01$, $r_0 = 0.0015$)

Left: $k = 2$; Right: $k = 20$