Scenario-based risk evaluation

Ruodu Wang

http://sas.uwaterloo.ca/~wang

Department of Statistics and Actuarial Science
University of Waterloo

Risk Management Institute, National University of Singapore
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Agenda

Based on some joint work with Damir Filipovic (Lausanne), Jie Shen (Waterloo), Yi Shen (Waterloo), Bin Wang (Beijing) and Johanna Ziegel (Bern)

Risk measures

A risk measure maps a risk (via a model) to a number

- regulatory capital calculation $\leftarrow$ our main interpretation
- management, optimization and decision making
- performance analysis and capital allocation
- pricing

Risks ...

- modelled by random losses in a specified period
- in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $p \in (0,1)$

$\mathrm{VaR}_p^{\mathbb{P}} : \mathcal{L} \to \mathbb{R}$,

$$\mathrm{VaR}_p^{\mathbb{P}}(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $p \in (0,1)$

$\mathrm{ES}_p^{\mathbb{P}} : \mathcal{L} \to \mathbb{R} \cup \{\infty\}$,

$$\mathrm{ES}_p^{\mathbb{P}}(X) = \frac{1}{1-p} \int_p^1 \mathrm{VaR}_q^{\mathbb{P}}(X)\mathrm{d}q.$$

- positive values of X represent losses
- $\mathcal{L}$: the set of all random variables in $(\Omega, \mathcal{A})$

Value-at-Risk and Expected Shortfall

The ongoing co-existence of VaR and ES:

- Basel III & IV - both
- Solvency II - VaR
- Swiss Solvency Test - ES
- US Solvency Framework (NAIC ORSA) - both

Law-based risk measures

VaR and ES are law-based (thus statistical risk functionals):
$\rho(X) = \rho(Y)$ if $X \stackrel{\mathrm{d}}{=}_{\mathbb{P}} Y$ (equal in distribution under $\mathbb{P}$)

- distributional models
- statistical inference and data analysis
- simulation tractability

However...

- Is the distribution alone enough to describe a risk, especially in a complex financial system?
- relation with economic scenarios?
- model uncertainty and robustness?

Law-based risk measures

Practical considerations

- statistical and simulation tractability
  - typically results in a law-based risk measure
- economic scenario sensitivity
  - typically results in a non-law-based risk measure
- model uncertainty and robustness
  - could be either law-based or not

Question

Can we incorporate the above considerations into a unified framework?

Basel's IMA ES formula for market risk

BIS (Bank for International Settlements) market risk formula:

[B16] Basel Committee on Banking Supervision: Standards, January 2016, *Minimum capital requirements for Market Risk*

The standard risk measure in [B16] for market risk is $\mathrm{ES}_{0.975}$.

- internal model approach (IMA): approved desks
- standard approach (SA): others

Basel's IMA ES formula for market risk

Aim: calculate capital charge for loss from a portfolio

- $T = 10d$ in [B16]
- $p = 0.975$, omitted below
- $X = \sum_{i=1}^{n} X_i$ is the aggregate portfolio loss at a given day
- Each risk factor is adjusted for their category of liquidity
- Two layers of further adjustments in p.52 - p.69 of [B16]

Basel's IMA ES formula for market risk

(i) Stress adjustment.

- Specify a set R of reduced risk factors which has a sufficiently long history of observation ("*at least span back to and including 2007*"), such that $\theta = \frac{\mathrm{ES}_F(X)}{\mathrm{ES}_R(X)} \le \frac{4}{3}$, where $\mathrm{ES}_F(X) = \mathrm{ES}_p(\sum_{i=1}^n X_i)$ and $\mathrm{ES}_R(X) = \mathrm{ES}_p(\sum_{i \in R} X_i)$.
- Compute $\mathrm{ES}_{R,S}(X)$, the ES of a model with the reduced risk factors, "*calibrated to the most severe 12-month period of stress*".
- Mathematically, for the collection $\mathcal{Q}$ of estimated models,

$$\mathrm{ES}_{R,S}(X) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q \left(\sum_{i \in R} X_i \right).$$

Basel's IMA ES formula for market risk

- Use the formula

$$\widetilde{\mathrm{ES}}(X) = \mathrm{ES}_{R,S}(X) \times \max\{\theta, 1\}$$

to get the stress-adjusted ES value,

- In particular, if the portfolio loss is modelled by only risk factors of sufficiently long history, then $R = \{1, \dots, n\}$ and

$$\widetilde{\mathrm{ES}}(X) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q\left(\sum_{i=1}^n X_i\right) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q(X).$$

Basel's IMA ES formula for market risk

(ii) Dependence adjustment.

- Risk factors are grouped into a range of broad regulatory risk classes
- Under the stressed scenario, compute the ES of each risk class, and take their sum $\widetilde{\mathrm{ES}}_C(X)$
  - equivalent to comonotonic ("non-diversified") risk classes
  - worst case dependence scenario
- Use the formula, for a constant λ $(= 0.5)$,

$$\mathrm{ES}(X) = \lambda\widetilde{\mathrm{ES}}(X) + (1-\lambda)\widetilde{\mathrm{ES}}_C(X)$$

- $\mathrm{ES}(X)$ is called the capital charge for modellable risk factors (IMCC)

Basel's IMA ES formula for market risk

In summary, the Basel's IMA ES formula[1]:

- Calculates ES of the same portfolio under different scenarios
  - stress (stressed, non-stressed)
  - dependence (diversified, non-diversified)
- These values are aggregated with mainly two operations (iteratively): maximum and linear combination

[1] In addition to (i) and (ii), the IMCC value will finally be adjusted by using the maximum of its present calculation and a moving average calculation of 60 days times a constant (currently 1.5).

Basel's IMA ES formula for market risk

Two other examples.

- The margin requirements calculation developed by the Chicago Mercantile Exchange
- The rating of a structured finance security is calculated according to its conditional distributions under each economic stress scenario (e.g. Standard and Poor's and Moody's).

Progress of the talk

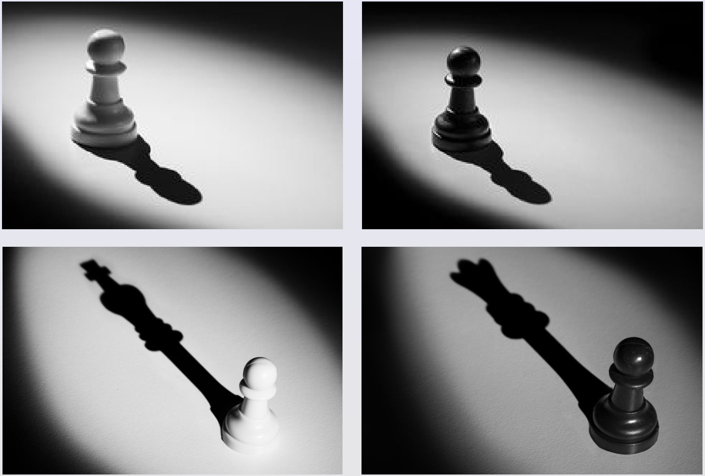
Scenario-based risk evaluation

Some metaphor

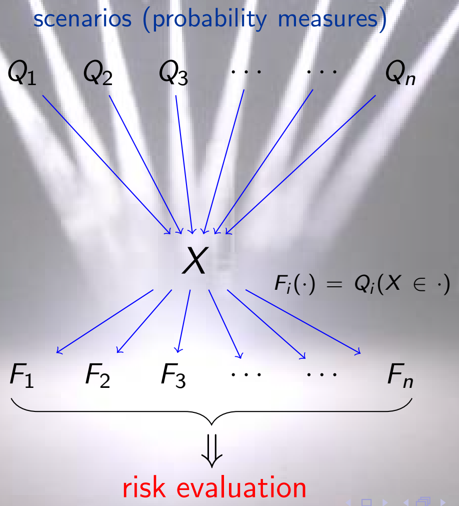
- Scenarios: light sources
- Risks (random outcomes): objects
- Distributions: shadows

Scenario-based risk evaluation

Scenario-based risk evaluation



Scenario-based risk evaluation



$\mathcal{Q}$-based risk measures

- Take a collection of scenarios $\mathcal{Q} \subset \mathcal{P}$ of interest
- $\mathcal{P}$: the set of probability measures (scenarios) on $(\Omega, \mathcal{A})$
- $\mathcal{X}$: a convex cone of random variables

Definition 1

A mapping ρ on $\mathcal{X}$ is *$\mathcal{Q}$-based* if $\rho(X) = \rho(Y)$ for $X, Y \in \mathcal{X}$ whenever $X \overset{\mathrm{d}}{=}_Q Y$ for all $Q \in \mathcal{Q}$.

- Two risks are equally risky if they are identically distributed under all scenarios of interest
- The scenarios should be pre-specified according to the application

Examples of $\mathcal{Q}$

- **(Basel formula)** In the [B16] ES formula, $\mathcal{Q}$ is the collection of many practical scenarios
- **(Economic scenario)** $\mathcal{Q} = \{\mathbb{P}(\cdot|\Theta = \theta) : \theta \in \mathbb{R}^d\}$, where Θ is an economic factor
- **(Robust evaluation)** $\mathcal{Q} = \{Q \in \mathcal{P} : d(Q, \mathbb{P}) \leq \delta\}$ where d is some statistical distance (e.g. Kullback-Leibler)
- **(Bayesian)** $\mathcal{Q} = \{Q_\theta : \theta \in \mathbb{R}^d\}$, a parametric family of models
- **(Simulation)** $\mathcal{Q} = \{\mathbb{P}_i : i = 1, \dots, N\}$, where $\mathbb{P}_i$ is the empirical measure of data or simulated sample
- **(Financial market)** $\mathcal{Q} = \{\mathbb{P}\} \cup \mathcal{Q}_M$ where $\mathcal{Q}_M$ is the set of martingale pricing measures in a financial market

$\mathcal{Q}$-based risk measures

$\mathcal{Q}$-based risk measures bridge law-based ones and generic ones

$$\underbrace{\{\mathbb{P}\}}_{\text{law-based}} \subset \underbrace{\mathcal{Q}}_{\mathcal{Q}\text{-based}} \subset \underbrace{\mathcal{P}}_{\text{generic}}$$

- Simplest $\mathcal{Q}$-based risk measures: taking an operation on some law-based risk measures under different scenarios

$\mathcal{Q}$-based risk measures

Example (Systemic risk measures).

For a fixed random variable S (the system) and $p \in (0,1)$, CoVaR is defined as:

$$\mathrm{CoVaR}_p^S(X) = \mathrm{VaR}_p^{\mathbb{P}}(S|X = \mathrm{VaR}_p^{\mathbb{P}}(X)), \quad X \in \mathcal{L},$$

and CoES is defined as:

$$\mathrm{CoES}_p^S(X) = \mathbb{E}^{\mathbb{P}}[S|S \geq \mathrm{CoVaR}_p^S(X)], \quad X \in \mathcal{L}.$$

CoVaR and CoES are determined by the joint distribution of (X, S), thus $\mathcal{Q}$-based risk measures for $\mathcal{Q} = \{\mathbb{P}(\cdot|S = s) : s \in \mathbb{R}\}$.

e.g. Adrian-Brunnermeier'16

$\mathcal{Q}$-based risk measures

Example.

The risk measure ρ based on multiple scenarios, given by

$$\rho(X) = \sup_{(w_1,\dots,w_n)\in\mathcal{W}} \left\{ \sum_{i=1}^{n} w_i \rho_{h_i}^{Q_i}(X) \right\}, \quad X \in \mathcal{L},$$

is a $\mathcal{Q}$-based risk measure for $Q \in \{Q_1, \dots, Q_n\}$.

Introduced in Section 3 of Kou-Peng'16

Max-ES and Max-VaR

Fix $\mathcal{Q}$ and $p \in (0,1)$.

Max-ES

The Max-ES (MES) is defined as

$$\mathrm{MES}_p^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q(X), \quad X \in \mathcal{L}.$$

Max-VaR

The Max-VaR (MVaR) is defined as

$$\mathrm{MVaR}_p^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{VaR}_p^Q(X), \quad X \in \mathcal{L}.$$

Max-ES and Max-VaR

Theorem 2

For $p \in (0,1)$, the following hold.

(i) $\mathrm{MES}_p^{\mathcal{Q}}$ *is coherent, but generally not comonotonic-additive.*

(ii) $\mathrm{MVaR}_p^{\mathcal{Q}}$ *is comonotonic-additive, but generally not coherent.*

Sharp contrast to the case of a single scenario!

Notes.

- a risk measure is coherent if it is monotone, translation-invariant, positively homogeneous and convex (or subadditive); this includes ES_p^Q
- a risk measure is comonotonic-additive if it is additive for comonotonic random variables; this includes ES_p^Q and VaR_p^Q

(Properties of risk measures: Artzner-Delbaen-Eber-Heath'99, Kusuoka'01)

Max-ES and Max-VaR

Alternative generalizations of ES ($\mathcal{Q} = \{Q_1, \dots, Q_n\}$):

(i) Average-ES

$$\mathrm{AES}_p^{\mathcal{Q}}(X) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{ES}_p^{Q_i}(X), \quad X \in \mathcal{L}.$$

(ii) integral Max-ES

$$\mathrm{iMES}_p^{\mathcal{Q}}(X) = \frac{1}{1-p}\int_p^1 \mathrm{MVaR}_q^{\mathcal{Q}}(X)\mathrm{d}q, \quad X \in \mathcal{L}.$$

(iii) replicated Max-ES

$$\mathrm{rMES}_p^{\mathcal{Q}}(X) = \mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n} F_{X,Q_i}^{-1}(U_i)\right), \quad X \in \mathcal{L},$$

where $U_1, \dots, U_n$ are iid $\mathrm{U}[0,1]$ under $\mathbb{P}$.

Max-ES and Max-VaR

Theorem 3

Let $\mathcal{Q}$ be a collection of n scenarios and $p \in (0,1)$.

(i) $\mathrm{AES}_p^{\mathcal{Q}}$ *is comonotonic-additive and coherent.*

(ii) $\mathrm{iMES}_p^{\mathcal{Q}}$ *is comonotonic-additive, but generally not coherent.*

(iii) $\mathrm{rMES}_p^{\mathcal{Q}}$ *is comonotonic-additive and coherent.*

(iv) $\mathrm{AES}_p^{\mathcal{Q}} \le \mathrm{MES}_p^{\mathcal{Q}} \le \mathrm{iMES}_p^{\mathcal{Q}} \le \mathrm{rMES}_p^{\mathcal{Q}}$ *on* $\mathcal{L}$.

(v) *If* $n = 1$, *then* $\mathrm{AES}_p^{\mathcal{Q}} = \mathrm{MES}_p^{\mathcal{Q}} = \mathrm{iMES}_p^{\mathcal{Q}} = \mathrm{rMES}_p^{\mathcal{Q}}$ *on* $\mathcal{L}$.

- After all, it is not clear which definition is the most natural generalization of ES.

$\mathcal{Q}$-based risk measures

Example. $\mathcal{Q} = \{Q_1, Q_2\}$,

$$\rho(X) = 2\mathbb{E}^{Q_1}[X] - \mathbb{E}^{Q_2}[X], \quad X \in \mathcal{X}.$$

- ρ is coherent $\Leftrightarrow$ $Q_2 \leq 2Q_1$, i.e. $\rho(X) = \mathbb{E}^{2Q_1 - Q_2}[X]$

Remarks.

- Properties of ρ depends on
  - how distributions are aggregated
  - relationship among $\mathcal{Q}$
- Many ways of aggregating distributions under each scenario
- Mathematical treatment is different from the law-based case
- New challenges!

Progress of the talk

Mathematical results

Mathematical contributions:

- Axiomatic characterization for all comonotonic-additive $\mathcal{Q}$-based risk measures
- Axiomatic characterization for all coherent $\mathcal{Q}$-based risk measures, if $\mathcal{Q}$ is mutually singular
- Equivalent condition for compatibility of scenarios with a given set of distributions

Assumptions:

- $\mathcal{X}$: the set of bounded random variables
- $\mathcal{Q} = \{Q_1, \dots . Q_n\}$: a finite set of scenarios

Distortion risk measures

- Q-distortion risk measure

  $\Leftrightarrow$ $\{Q\}$-based, comonotonic-additive, monetary
- Q-spectral risk measure

  $\Leftrightarrow$ $\{Q\}$-based, comonotonic-additive, coherent

They have a Choquet integral form, for $X \geq 0$,

$$\rho_g(X) = \int X \mathrm{d} g \circ Q = \int_0^\infty g \circ Q(X > x) \mathrm{d}x.$$

The distortion function $g : [0, 1] \to [0, 1]$, $g(0) = 1 - g(1) = 0$,

- g increasing $\Leftrightarrow$ ρ_g distortion
- g increasing and concave $\Leftrightarrow$ ρ_g spectral

(Yaari'87, Wang-Young-Panjer'97, Kusuoka'01, Föllmer-Schied'16)

$\mathcal{Q}$-distortion risk measures

$\mathcal{Q}$-distortion risk measures

- $\mathcal{Q}$-distortion risk measure

  $\Leftrightarrow$ $\mathcal{Q}$-based, comonotonic-additive, monetary

- $\mathcal{Q}$-spectral risk measure

  $\Leftrightarrow$ $\mathcal{Q}$-based, comonotonic-additive, coherent

They have a Choquet integral form, for $X \geq 0$,

$$\rho_\psi(X) = \int_0^\infty \psi \circ (Q_1, \dots, Q_n)(X > x)\mathrm{d}x.$$

The distortion function $\psi : [0,1]^n \to [0,1]$, $\psi(\mathbf{0}) = 1 - \psi(\mathbf{1}) = 0$,

- ψ componentwise increasing $\Rightarrow$ ρ_ψ distortion
- ψ componentwise increasing, componentwise concave and submodular $\Rightarrow$ ρ_ψ spectral

$\mathcal{Q}$-distortion risk measures

Example.

$$2\mathbb{E}^{Q_1}[X] - \mathbb{E}^{Q_2}[X] = \int X \mathrm{d}\psi \circ (Q_1, Q_2)$$

where $\psi(u_1, u_2) = 2u_1 - u_2$, $(u_1, u_2) \in [0, 1]^2$. If $Q_2 \leq 2Q_1$, then ρ is a $\mathcal{Q}$-spectral risk measure, but ψ is not componentwise increasing.

- A characterization of ψ for $\mathcal{Q}$-distortion risk measures is available depending on $\mathcal{Q}$.
- One can choose $\psi(\mathbf{u}) = 1 - C(1 - \mathbf{u})$ for some copula C

Coherent $\mathcal{Q}$-based risk measures

Theorem 4

Suppose that $\mathcal{Q}$ is mutually singular. A functional $\rho : \mathcal{X} \to \mathbb{R}$ is a $\mathcal{Q}$-based coherent risk measure if and only if it is the maximum of some mixtures of Q-Expected Shortfalls, $Q \in \mathcal{Q}$.

- mutual singularity is used twice: once to establish the Fatou property, once to show that a Kusuoka-type argument leads to spectral risk measures
- Characterization is unclear if $\mathcal{Q}$ is not mutually singular

Basel's IMA ES formula

The Basel's IMA ES formula is roughly a combination of maximums and linear combinations of ES under various scenarios.

Compare with Theorem 4: not too bad, after all!

Progress of the talk

Non-parametric statistical inference

Suppose observations are available under $\mathbb{P}$, and we aim to calculate a $\mathcal{Q}$-based risk measure evaluated at X, e.g. $\mathrm{MES}_p^{\mathcal{Q}}(X)$.

- Only observations of X under $\mathbb{P}$ are not enough
- We need a framework to allow for inference of $\mathrm{d}Q/\mathrm{d}\mathbb{P}$ for $Q \in \mathcal{Q}$.
- Think about the Basel evaluation procedure: each observation is observed together with the economic scenarios

Statistical inference

General settings

(i) **Full model.** Vector data $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are observed, where $\mathbf{X}_j = (X_j, Z_{1,j}, \ldots, Z_{n,j})$, $j = 1, \ldots, N$, are observations from $(X, \frac{\mathrm{d}Q_1}{\mathrm{d}\mathbb{P}}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}\mathbb{P}})$ under $\mathbb{P}$.

(ii) **Categorical model.** Suppose that $Q_i(\cdot) = \mathbb{P}(\cdot|A_i)$ for some $A_i \in \mathcal{F}$ with $\mathbb{P}(A_i) > 0$, $i = 1, \ldots, n$. Vector data $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are observed, where $\mathbf{X}_j = (X_j, Z_{1,j}, \ldots, Z_{n,j})$, $j = 1, \ldots, N$, are observations from $(X, \mathrm{I}_{A_1}, \ldots, \mathrm{I}_{A_n})$ under $\mathbb{P}$.

(iii) **Individual models.** n sequences of data $\{X_1^1, \ldots, X_{N_1}^1\}, \ldots, \{X_1^n, \ldots, X_{N_n}^n\}$ are observed, where for $i = 1, \ldots, n$, $X_1^i, \ldots, X_{N_i}^i$ are observations of X under Q_i.

Empirical distributions

The empirical distribution $\hat{F}_N^{Q_i}$ of X under Q_i, $i = 1, \dots, n$:

(i) **Full model.**

$$\hat{F}_N^{Q_i}(x) = \frac{1}{N} \sum_{j=1}^{N} Z_{i,j} \mathrm{I}_{\{X_j \le x\}}, \quad x \in \mathbb{R}.$$

(ii) **Categorical model.**

$$\hat{F}_N^{Q_i}(x) = \frac{\sum_{j=1}^{N} Z_{i,j} \mathrm{I}_{\{X_j \le x\}}}{\sum_{j=1}^{N} Z_{i,j}}, \quad x \in \mathbb{R}.$$

(iii) **Individual models.** $(N = \sum_{i=1}^{n} N_i)$

$$\hat{F}_N^{Q_i}(x) = \frac{1}{N_i} \sum_{j=1}^{N_i} \mathrm{I}_{\{X_j^i \le x\}}, \quad x \in \mathbb{R}.$$

Empirical estimators

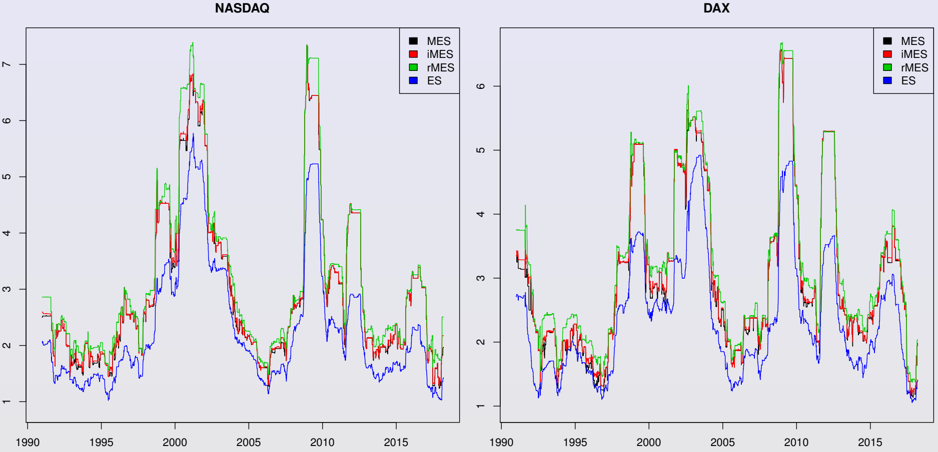
- Under regular assumptions on the observations, consistency of $F_N^{Q_i}$ for F_{X,Q_i}, i.e. $F_{Q_i}^N \to F_{X,Q_i}$ can be established in each setting
- One can empirically estimate a $\mathcal{Q}$-based risk measure by applying it to the empirical distributions
- Consistency and asymptotics of such empirical estimators are possible, under suitable assumptions on both the observations and the risk measure

$\mathcal{Q}$-based Expected Shortfalls for economic scenarios

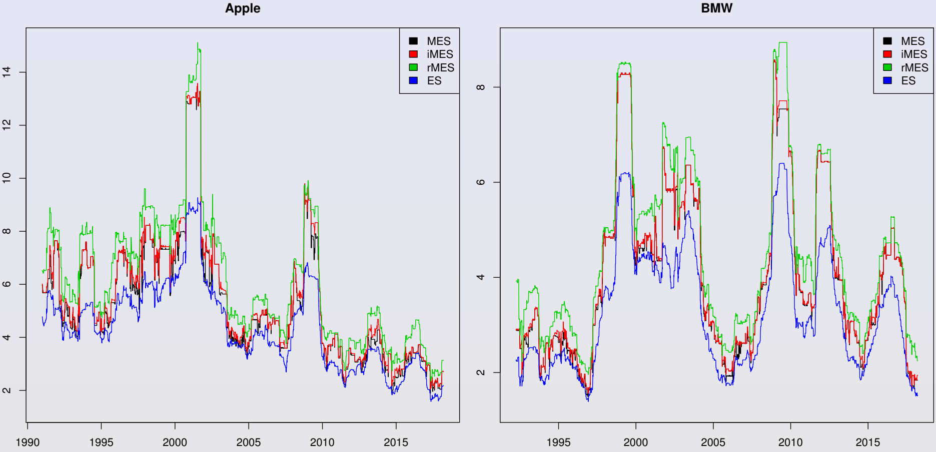
Simple empirical study

- A series of returns $(X_t)_{t\in\mathbb{N}}$ for each trading day
- Rolling window of length 250 for the estimation
- $n = 4$ scenarios taken from $\{\text{high volatility}, \text{low volatility}\} \times \{\text{good economy}, \text{bad economy}\}$.
- Divide the data into 4 categories according to first VIX (since 1990) and then S&P 500.
- $Q_i = \mathbb{P}(\cdot|\Theta = \theta_i)$.
- Estimate the risk measures $\mathrm{ES}_p^{\mathbb{P}}$, $\mathrm{MES}_p^{\mathcal{Q}}$, $\mathrm{iMES}_p^{\mathcal{Q}}$, and $\mathrm{rMES}_p^{\mathcal{Q}}$
- Take $p = 0.9$

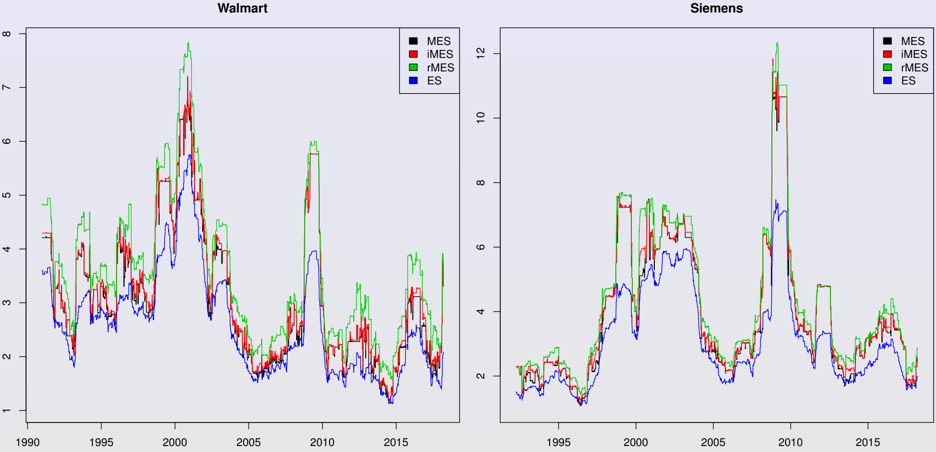
$\mathcal{Q}$-based Expected Shortfalls for economic scenarios



$\mathcal{Q}$-based Expected Shortfalls for economic scenarios



$\mathcal{Q}$-based Expected Shortfalls for economic scenarios



$\mathcal{Q}$-based Expected Shortfalls for economic scenarios

Observations

- MES_p and iMES_p yield similar values.
- During times of financial stress, MES_p and ES_p deviate substantially.
- For the indices MES_p and rMES_p are closer than for the stock returns.
- During economically stable periods, the ratio between rMES_p and MES_p is generally larger than during financial stress.
- The ratio $\mathrm{MES}_p/\mathrm{ES}_p$ distinguishes the early 2000s recession from the 2008 financial crisis, except for Apple.

The Basel stress-adjustment for Expected Shortfall

- P_t^i, $i = 1, \dots, n$, $t \in \mathbb{N}$ denote the time-t price of security i
- $X_t^i = -(P_t^i / P_{t-1}^i - 1)$ the daily negative return
- Construct a portfolio with price process $V_t = \sum_{i=1}^n \alpha_i P_t^i$ where α_i is fixed throughout the investment period
- Each portfolio starts from \$1
- At time $t-1$, the numbers α_i and P_{t-1}^i are known
- Calculate ES of the daily loss $V_{t-1} - V_t$

$$\mathrm{ES}_p^P(V_{t-1} - V_t) = \mathrm{ES}_p^P\left(\sum_{i=1}^n X_t^i \alpha_i P_{t-1}^i\right),$$

where $p = 0.975$ in [B16]

The Basel stress-adjustment for Expected Shortfall

- Stress-adjustment: to mimic [B16] (dating back to 2007), we date back to 10 years for all t
- Evaluate

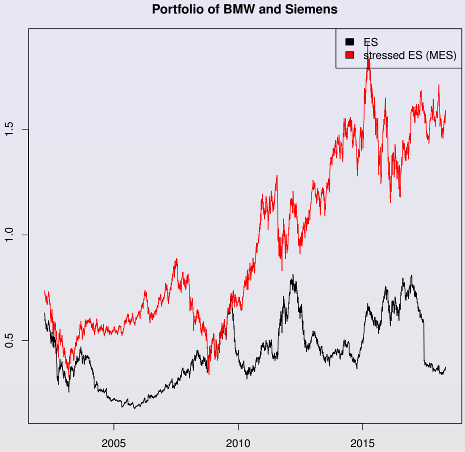
$$\mathrm{MES}_p^{\mathcal{Q}}(V_{t-1} - V_t) = \max_{j=1,\dots,N} \mathrm{ES}_p^{Q_j}\left(\sum_{i=1}^{n} X_t^i \alpha_i P_{t-1}^i\right),$$

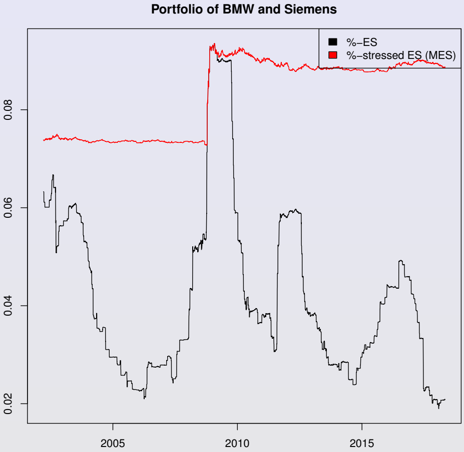
where $N = 2251$, $\mathcal{Q} = \{Q_j\}_{j=1,\dots,N}$, and under Q_j, $(X_t^1, \dots, X_t^n)$ is distributed according to its empirical distribution over the time period $[t-j-249, t-j]$.

- A US stocks portfolio (Apple and Walmart) and a German stocks portfolio (BMW and Siemens)

The Basel stress-adjustment for Expected Shortfall

The Basel stress-adjustment for Expected Shortfall





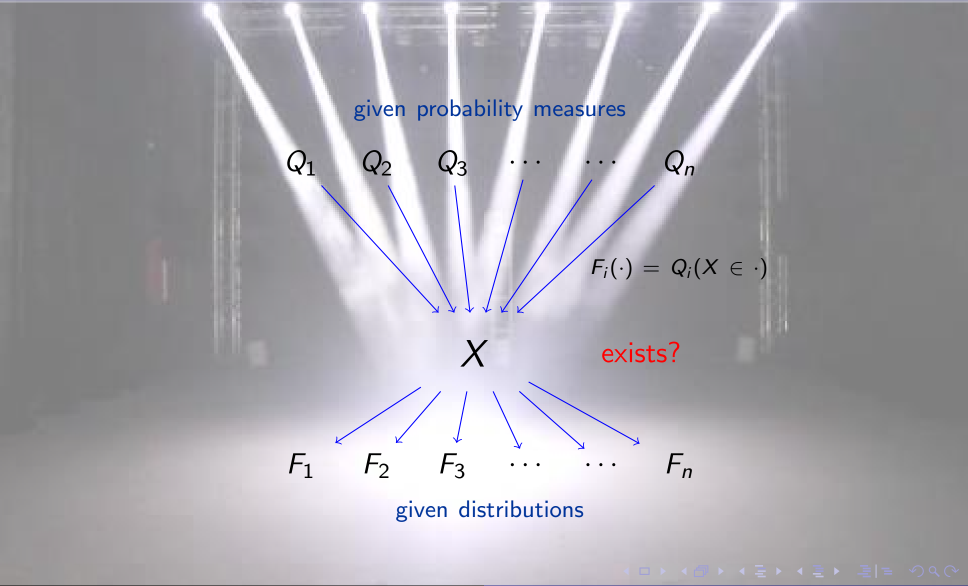
The Basel stress-adjustment for Expected Shortfall

Observations

- The percentage MES is relatively stable (6% - 9%), and the ES is changing drastically (2% - 9%).
- The US portfolio has a high percentage MES till 1998 (because of the Black Monday, Oct 19, 1987).
- Right before 2007:
  - Using ES, both portfolio exhibit serious under capitalization, and their ES values increased drastically when the financial crisis took place.
  - Using MES, the requirement of capital for both portfolios only increased moderately during the financial crisis.

Progress of the talk

Compatibility of scenarios



Compatibility of scenarios

Notation

- $\mathcal{F}$: the set of distribution measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- $\mathcal{J}$: a (possibly uncoutable) set of indices.

Definition 5

$(Q_i)_{i\in\mathcal{J}} \subset \mathcal{P}$ and $(F_i)_{i\in\mathcal{J}} \subset \mathcal{F}$ are compatible if there exists a random variable X in $(\Omega, \mathcal{A})$ such that $X \sim_{Q_i} F_i$ for $i \in \mathcal{J}$.

Compatibility of scenarios

Simple intuitions

- $Q_1, \dots, Q_n$ identical $\Rightarrow F_1, \dots, F_n$ identical
- $Q_1, \dots, Q_n$ equivalent $\Rightarrow F_1, \dots, F_n$ equivalent
- $Q_1, \dots, Q_n$ mutually singular $\Rightarrow F_1, \dots, F_n$ arbitrary
- $F_1, \dots, F_n$ mutually singular $\Rightarrow Q_1, \dots, Q_n$ mutually singular

$Q_1, \dots, Q_n$ are more different (heterogeneous) than $F_1, \dots, F_n$!

- How do we model heterogeneity of $(Q_i)_{i \in \mathcal{J}}$ and $(F_i)_{i \in \mathcal{J}}$?

Characterization of compatibility via heterogeneity

Theorem 6

Suppose that $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$ is conditionally atomless. $(Q_1, \ldots, Q_n)$ and $(F_1, \ldots, F_n) \in \mathcal{F}^n$ are compatible if and only if

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \ldots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\Big|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Big|_Q$$

for some $F \in \mathcal{F}$, $(F_1, \ldots, F_n) \ll F$ and $Q \in \mathcal{P}$, $(Q_1, \ldots, Q_n) \ll Q$, where $\prec_{\mathrm{cx}}$ is the multivariate convex order.

- We call this relation heterogeneity order

more technical details

$(Q_1, \ldots, Q_n)$ is conditionally atomless if there exist $Q \in \mathcal{P}$, $(Q_1, \ldots, Q_n) \ll Q$ and a continuous random variable independent of $(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q})$ under Q.

Characterization of compatibility via heterogeneity

Remarks

- Easy to check, especially for $n = 2$
- It is insufficient to assume each $(\Omega, \mathcal{A}, Q_i)$ is atomless
- The result can be generalized to measures on a general measurable space, e.g. measures on $\mathbb{R}^d$ or on the path space of càdlàg processes

There is a deep connection between heterogeneity order and the comparison of experiments in statistical decision theory (e.g. Blackwell'51).

Relation to the Girsanov Theorem

- $P \in \mathcal{P}$ and $B = \{B_t\}_{t\in[0,T]}$ is a P-standard BM
- θ and μ are $[0, T]$-square integrable deterministic processes
- Q_θ is given by

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}P} = e^{\int_0^T \theta_t \mathrm{d}B_t - \frac{1}{2}\int_0^T \theta_t^2 \mathrm{d}t}$$

- G_μ is the distribution measure of a BM with drift process μ

Theorem 7

Suppose that $\mu_t \neq 0$ for a.e. $t \in [0, T]$. (P, Q_θ) and (G_0, G_μ) are compatible if and only if $\int_0^T \mu_t^2 \mathrm{d}t \le \int_0^T \theta_t^2 \mathrm{d}t$.

Related optimization problems

Given $P, Q \in \mathcal{P}$,

- If $X \sim_Q G$, find the maximum and minimum values of $\mathbb{E}^P[X]$, $\mathrm{VaR}_p^P(X)$, $\mathrm{ES}_p^P(X)$, $\mathrm{Var}^P(X)$, ...
- What if we know $X \sim_{Q_i} F_i$, $i = 1, \dots, n$? (very challenging)
- Connected to many well-known problems, e.g. the knapsack problem (continuous setting), robust utility, robust variance, Fréchet-Hoeffding, Neyman-Pearson, ...

Progress of the talk

Summary

- A framework for scenario-based risk evaluation
  - new classes of risk measures (e.g. MES and MVaR)
  - axiomatic characterization
  - statistical analysis
  - characterization of compatibility
  - applicable to almost all existing distributional problems
- Related mathematics:
  - functional analysis (e.g. Hahn-Banach, Meyer-Choquet)
  - vector measure theory (e.g. Lyapunov)
  - statistical decision theory (e.g. Blackwell)
  - dependence modeling (e.g. Fréchet-Hoeffding)
  - many open mathematical and optimization questions!

Allegory

"How could they see anything but the shadows if they were never allowed to move their heads?"

- Plato, *Republic* (380 BC), The Allegory of the Cave

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Multivariate convex order

Fix a positive integer n.

- For $\mathbf{X} \in L_1^n(\Omega_1, \mathcal{A}_1, P_1)$ and $\mathbf{Y} \in L_1^n(\Omega_2, \mathcal{A}_2, P_2)$, write $\mathbf{X}|_{P_1} \prec_{\mathrm{cx}} \mathbf{Y}|_{P_2}$, if $\mathbb{E}^{P_1}[f(\mathbf{X})] \le \mathbb{E}^{P_2}[f(\mathbf{Y})]$ for all convex functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the sets of probability measures on two arbitrary measurable spaces.

Heterogeneity order

Definition 8 (Heterogeneity order)

$(P_1, \dots, P_n) \in \mathcal{M}_1^n$ is dominated by $(Q_1, \dots, Q_n) \in \mathcal{M}_2^n$ in heterogeneity, denoted by $(P_1, \dots, P_n) \prec_{\mathrm{h}} (Q_1, \dots, Q_n)$, if

$$\left(\frac{\mathrm{d}P_1}{\mathrm{d}P}, \dots, \frac{\mathrm{d}P_n}{\mathrm{d}P}\right)\bigg|_P \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \dots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\bigg|_Q .$$

for some $P \in \mathcal{M}_1$ and $Q \in \mathcal{M}_2$ with $(P_1, \dots, P_n) \ll P$ and $(Q_1, \dots, Q_n) \ll Q$.

- $\prec_{\mathrm{h}}$ is easy to check, especially for $n = 2$

Properties of heterogeneity order

The reference measures P and Q in

$$\left(\frac{\mathrm{d}P_1}{\mathrm{d}P}, \ldots, \frac{\mathrm{d}P_n}{\mathrm{d}P}\right)\Bigg|_P \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q \qquad (1)$$

do not matter, and therefore $\prec_{\mathrm{h}}$ is a partial order.

Lemma 9

For $(P_1, \ldots, P_n) \in \mathcal{M}_1^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{M}_2^n$, equivalent are:

(i) $(P_1, \ldots, P_n) \prec_{\mathrm{h}} (Q_1, \ldots, Q_n)$.

(ii) *For $P = \frac{1}{n}\sum_{i=1}^n P_i$ and $Q = \frac{1}{n}\sum_{i=1}^n Q_i$, (1) holds.*

(iii) *For any $Q \in \mathcal{M}_2^*$, there exists $P \in \mathcal{M}_1^*$ such that (1) holds.*

Properties of heterogeneity order

For $(P_1, \ldots, P_n) \in \mathcal{M}_1^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{M}_2^n$,

(i) $P_1, \ldots, P_n$ identical $\Rightarrow (P_1, \ldots, P_n) \prec_{\mathrm{h}} (Q_1, \ldots, Q_n)$;

(ii) $Q_1, \ldots, Q_n$ mutually singular $\Rightarrow (P_1, \ldots, P_n) \prec_{\mathrm{h}} (Q_1, \ldots, Q_n)$.

If $(P_1, \ldots, P_n) \prec_{\mathrm{h}} (Q_1, \ldots, Q_n)$,

(iii) $Q_1, \ldots, Q_n$ identical $\Rightarrow P_1, \ldots, P_n$ identical;

(iv) $Q_1, \ldots, Q_n$ equivalent $\Rightarrow P_1, \ldots, P_n$ equivalent;

(v) $P_1, \ldots, P_n$ mutually singular $\Rightarrow Q_1, \ldots, Q_n$ mutually singular.

(ii) is due to the Meyer-Choquet Theorem

Heterogeneity order and compatibility

Lemma 10

If $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$ *and* $(F_1, \ldots, F_n) \in \mathcal{F}^n$ *are compatible, then* $(F_1, \ldots, F_n) \prec_{\mathrm{h}} (Q_1, \ldots, Q_n)$.

- why are heterogeneity order and compatibility closely related?

Second characterization of compatibility

Theorem 11

For $(Q_i)_{i\in\mathcal{J}} \subset \mathcal{P}$ and $(F_i)_{i\in\mathcal{J}} \subset \mathcal{F}$ and $X \in L(\Omega, \mathcal{A})$, assuming that there exists a probability measure in $\mathcal{P}$ dominating $(Q_i)_{i\in\mathcal{J}}$, equivalent are:

(i) $X \sim_{Q_i} F_i$ *for all* $i \in \mathcal{J}$.

(ii) *For all $Q \in \mathcal{P}$ dominating $(Q_i)_{i\in\mathcal{J}}$, the probability measure $F = Q \circ X^{-1}$ dominates $(F_i)_{i\in\mathcal{J}}$, and*

$$\frac{\mathrm{d}F_i}{\mathrm{d}F}(X) = \mathbb{E}^Q\left[\frac{\mathrm{d}Q_i}{\mathrm{d}Q}\middle|X\right] \quad \text{for all } i \in \mathcal{J}. \tag{2}$$

(iii) *For some $Q \in \mathcal{P}$ dominating $(Q_i)_{i\in\mathcal{J}}$, the probability measure $F = Q \circ X^{-1}$ dominates $(F_i)_{i\in\mathcal{J}}$ and (2) holds.*

Special case: $n = 2$

Corollary 12

For $(Q_1, Q_2) \in \mathcal{P}^2$, $Q_1 \ll Q_2$ and $(F_1, F_2) \in \mathcal{F}^2$, (Q_1, Q_2) and (F_1, F_2) are compatible if and only if there exists $X \in L(\Omega, \mathcal{A})$ with distribution F_2 under Q_2, such that $F_1 \ll F_2$ and

$$\frac{\mathrm{d}F_1}{\mathrm{d}F_2}(X) = \mathbb{E}^{Q_2}\left[\frac{\mathrm{d}Q_1}{\mathrm{d}Q_2}\middle| X\right].$$

- Such conditions are not easy to check in general

For relation between convex order and conditional expectations, see e.g. Theorem 3.4.2 of Müller-Stoyan 2002; Theorem 7.A.1, Shaked-Shanthikumar 2007

Second characterization of compatibility

Is $(F_1, \dots, F_n) \prec_{\mathrm{h}} (Q_1, \dots, Q_n)$ sufficient for compatibility?

From

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \dots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \dots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\bigg|_Q$$

to the existence of X such that

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \dots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)(X) = \mathbb{E}^Q\left[\left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \dots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\bigg|X\right].$$

is a martingale construction problem in the same probability space.

---

e.g. Hirsch-Profeta-Roynette-Yor 2011 (martingale construction for *peacocks*)

Conditionally atomless measures

Definition 13

$(Q_1, \dots, Q_n) \in \mathcal{P}^n$ is conditionally atomless if there exist $Q \in \mathcal{P}$ dominating $(Q_1, \dots, Q_n)$ and a continuous random variable in $(\Omega, \mathcal{A})$ independent of $(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \dots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q})$ under Q.

- Q can always be chosen as $\frac{1}{n}\sum_{i=1}^n Q_i$.
- If $(Q_1, \dots, Q_n)$ is conditionally atomless, then each of $(\Omega, \mathcal{A}, Q_i)$, $i = 1, \dots, n$, is atomless.
- If $Q_1, \dots, Q_n$ are mutually singular and each of $(\Omega, \mathcal{A}, Q_i)$, $i = 1, \dots, n$, is atomless, then $(Q_1, \dots, Q_n)$ is conditionally atomless.

Characterization of compatibility via heterogeneity

Assume $(F_1, \dots, F_n) \ll F$ and $(Q_1, \dots, Q_n) \ll Q$. A key condition

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \dots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\Bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \dots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q . \tag{3}$$

Theorem 14

Suppose that $(Q_1, \dots, Q_n) \in \mathcal{P}^n$ is conditionally atomless, and $(F_1, \dots, F_n) \in \mathcal{F}^n$. Equivalent are

(i) *$(Q_1, \dots, Q_n)$ and $(F_1, \dots, F_n)$ are compatible.*

(ii) *For some $F \in \mathcal{F}$ and $Q \in \mathcal{P}$, (3) holds.*

(iii) *For $F = \frac{1}{n}\sum_{i=1}^n F_i$ and $Q = \sum_{i=1}^n Q_i$, (3) holds.*

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