# Competitive Equilibria in a Comonotone Market

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Based on joint work with Tim Boonen (Amsterdam) and Fangda Liu (CUFE, Beijing)

# The market

A one-period exchange market is described by a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a set of bounded random future wealths $\mathcal{X}$.

- There are $n$ agents and $N = \{1, \ldots, n\}$. Each of them is endowed with an endowment $\xi_i \in \mathcal{X}$ and uses an objective functional $V_i : \mathcal{X} \to \mathbb{R}$ to model his preference.
- The total future wealth is $X = \sum_{i=1}^{n} \xi_i$, and its range $R(X) \subset \mathbb{R}$ is an interval.
- The current price of a random wealth $Y \in \mathcal{X}$ is given by $\mathbb{E}^{\mathbb{Q}}[Y]$ for some pricing measure $\mathbb{Q} \in \mathcal{P}$, where $\mathcal{P}$ is the set of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

$\mathbb{Q}$ will be an output of the market equilibrium.

# Competitive Equilibria (Arrow-Debreu Equilibria)

In an equilibrium, aggregate supplies will equal aggregate demands for every market state.

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## Definition

An allocation $(X_1, \ldots, X_n) \in \mathcal{X}^n$ and a pricing measure $\mathbb{Q} \in \mathcal{P}$ constitute an (Arrow-Debreu) competitive equilibrium if

- For $i \in N$, $X_i$ satisfies the budget constraint: $\mathbb{E}^{\mathbb{Q}}[X_i] \le \mathbb{E}^{\mathbb{Q}}[\xi_i]$
- For $i \in N$, $X_i$ maximizes the agent's objective:

$$V_i(Y) \le V_i(X_i), \qquad \text{for all } Y \in \mathcal{X} \text{ and } \mathbb{E}^{\mathbb{Q}}[Y] \le \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

- The market is cleared:

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \xi_i.$$

In a complete market, the set of admissible allocations is

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}.$$

Competitive equilibria in a complete market:

- Early work (expected utility): Arrow-Debreu'54, Borch'62
- Cumulative perspective theory: De Gorgi-Hens-Rieger'10
- Concave dual utility: Garlier-Dana'08, Dana'11, Boonen'15
- Rank dependent utility: Xia-Zhou'16, Jin-Xia-Zhou'18

For objectives other than expected utilities, finding competitive equilibria is a generally very challenging question

# Comonotonicity

## Definition

A random vector $(Y_1, \dots, Y_n)$ is comonotonic if

$$(Y_1, \dots, Y_n) = (f_1(Y), \dots, f_n(Y)),$$

holds for some non-decreasing functions $f_1, \dots, f_n$ and a random variable $Y$.

- $Y$ can be chosen as $\sum_{i=1}^n Y_i$; e.g. Denneberg'94.
- $(Y_1, Y_2)$ is counter-monotonic if $(-Y_1, Y_2)$ is comonotonic.

## Known results.

In a complete market, when agents have the same belief $\mathbb{P}$, under mild conditions, a *competitive equilibrium* $((X_1^*, \ldots, X_n^*), \mathbb{Q}) \in \mathbb{A}_n(X) \times \mathcal{P}$ satisfies

i. $(X_1^*, \ldots, X_n^*)$ is comonotonic.
ii. $(X_i^*, \eta)$ is counter-monotonic.
iii. $(X, \eta)$ is counter-monotonic, where $\eta$ is the *pricing kernel*

$$\eta = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$$

## Obsevation.

A complete market leads to comonotonic allocations, which are counter-monotone with the pricing kernel.

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## Obsevation.

A complete market leads to comonotonic allocations, which are counter-monotone with the pricing kernel.

## Question.

What happens if we constrain the feasible set of allocations to be comonotonic in the first place?

# Insurance

- In an insurance policy, the underlying risk is $Y$.
- $Y$ is shared by a policyholder and several insurers.
- To avoid *Moral Hazard*, no one should have the incentive to hope for a larger loss.
- *Slow growth property.* For the policyholder, the ceded part $f(Y)$ should be comonotonic with the retained part $Y - f(Y)$, or equivalently

$$0 \leq f(x) - f(y) \leq x - y, \quad 0 \leq y \leq x.$$

# Comonotone Market

- Allocations are constrained in the set

$$C(X) = \{Y \in \mathcal{X} : (Y, X - Y) \text{ is comonotonic }\}.$$

Thus $Y \in C(X)$ if and only if $Y = f(X)$ for some $f \in \mathcal{F}$, where

$$\mathcal{F} = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continuous and a.e. differentiable,} \\ 0 \leq f'(z) \leq 1 \text{ for } z \in \mathbb{R} \end{array} \right\}.$$

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- The set of admissible allocations if

$$\mathbb{A}_n^c(X) = \left\{ (X_1, \ldots, X_n) \in (C(X))^n : \sum_{i=1}^{n} X_i = X \right\}.$$

# Comonotone Market

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- The set of admissible allocations if

$$\mathbb{A}_n^c(X) = \left\{ (X_1, \ldots, X_n) \in (C(X))^n : \sum_{i=1}^n X_i = X \right\}.$$

- In the comonotone market, a *competitive equilibrium* is a pair $((X_1, \ldots, X_n), \mathbb{Q}) \in \mathbb{A}_n^c(X) \times \mathcal{P}$ such that $\mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i]$ and

$$V_i(X_i) = \max \left\{ V_i(Y_i) : Y_i \in C(X),\ \mathbb{E}^{\mathbb{Q}}[Y_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i] \right\}, \quad i \in N.$$

For the same set of agents

Comonotone market $\subsetneq$ Complete market

For the same set of agents

| | | |
|---|---|---|
| Comonotone market | $\subsetneq$ | Complete market |
| $\downarrow$ | | $\downarrow$ |
| Constrained CE (CCE) | | Unconstrained CE (UCE) |

For the same set of agents

$$\begin{array}{ccc} \text{Comonotone market} & \subsetneq & \text{Complete market} \\ \downarrow & & \downarrow \\ \text{Constrained CE (CCE)} & \Leftarrow & \text{Unconstrained CE (UCE)} \end{array}$$

Some results.

- A UCE is always a CCE (under some mild conditions).

For the same set of agents

| Comonotone market | $\subsetneq$ | Complete market |
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| $\downarrow$ | | $\downarrow$ |
| Constrained CE (CCE) | $\Leftarrow$ | Unconstrained CE (UCE) |

Some results.

- A UCE is always a CCE (under some mild conditions).
- A CCE is not necessarily a UCE.
- In a CCE $((X_1^*, \ldots, X_n^*), \mathbb{Q})$, $X_i^*$, $i \in N$ and $X$ may not be counter-monotonic with $\eta = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$. Thus, a sharp contrast to the case of complete market.

# Pareto-optimality

## Definition (Pareto-optimal allocations)

Fix objective functionals $V_1, \dots, V_n$, total wealth $X \in \mathcal{X}$ and initial endowments $\xi_1, \dots, \xi_n \in \mathcal{X}$.

(i) In the comonotone market, an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n^c(X)$ is *Pareto-optimal* if for any allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n^c(X)$, $V_i(Y_i) \geq V_i(X_i)$ for $i \in N$ implies $V_i(Y_i) = V_i(X_i)$ for $i \in N$.

(ii) In the complete market, an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is *Pareto-optimal* if for any allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$, $V_i(Y_i) \geq V_i(X_i)$ for $i \in N$ implies $V_i(Y_i) = V_i(X_i)$ for $i \in N$.

# Dual utility

The set of *distortion functions*

$$\mathcal{G} = \left\{ g : [0,1] \to [0,1] \;\middle|\; \begin{array}{l} g \text{ is continuous and increasing,} \\ g(0) = 0 \text{ and } g(1) = 1 \end{array} \right\}.$$

## Definition

A *dual utility (DU)* functional $D_g$ with distortion function $g \in \mathcal{G}$ is defined as a *Choquet integral*, namely, for $Y \in \mathcal{X}$,

$$D_g(Y) = \int Y \mathrm{d}\,(g \circ \mathbb{P}) := \int_{-\infty}^{0} (g(S_Y(z)) - 1)\,\mathrm{d}z + \int_{0}^{\infty} g(S_Y(z))\mathrm{d}z.$$

References: Yaari'87, Denneberg'94, Wang-Panjer-Young'97

# Rank dependent utility

## Definition

For an increasing function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and a distortion function $g \in \mathcal{G}$, a *rank-dependent utility (RDU)* functional $R_{u,g}$ is given by

$$R_{u,g}(Y) = D_g(u(Y)) = \int u(Y)\mathrm{d}(g \circ \mathbb{P}), \quad Y \in \mathcal{X}.$$

- $R_{u,g}$ is consistent with strong risk aversion if and only if $u$ is concave and $g$ is convex.
- The expected utility functional (EU) is a special case of RDU when $g(x) = x$ for $x \in [0, 1]$.
- The DU is a special case of an RDU when $u(x) = x$ for $x \in \mathbb{R}$.

# Competitive equilibria with dual utilities

In a comonotone market, where agents are equipped with dual utilities, we investigate following issues.

- Solving the individual optimization.
- Existence and the close form of a competitive equilibrium.
- Fundamental theorems of welfare economics.

# DU-comonotone market

Individual optimization:
Each agent is to find $X_i^*$ which solves

$$\max_{X_i \in C(X)} V_i(X_i) = D_g(X_i) = \int_\Omega X_i \mathrm{d}\left(g_i \circ \mathbb{P}\right), \qquad (1)$$
$$\text{s.t.} \quad \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

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Each agent is to find $X_i^*$ which solves

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$$\text{s.t.} \quad \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

## Proposition

*For a fixed $\mathbb{Q}$ and $f_i \in \mathcal{F}$, the random variable $X_i^* = f_i(X)$ solves (1) if and only if for a.e. $z \in R(X)$,*

$$f_i'(z) = \begin{cases} 1, & \text{if } g_i(S_X(z)) > \mathbb{Q}(X > z), \\ 0, & \text{if } g_i(S_X(z)) < \mathbb{Q}(X > z). \end{cases} \tag{2}$$

# Existence of CCE

## Theorem (1)

*In the DU-comonotone market, the following holds:*

(i) *A competitive equilibrium always exists.*

# Existence of CCE

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*In the DU-comonotone market, the following holds:*

(i) *A competitive equilibrium always exists.*

(ii) *The pair* $((X_1^*, \dots, X_n^*), \mathbb{Q})$ *is a competitive equilibrium if and only if*

(a) $g_{N,2}(S_X(z)) \leq \mathbb{Q}(X > z) \leq g_{N,1}(S_X(z))$ *for* $z \in R(X)$*, where* $g_{N,1}$ *and* $g_{N.2}$ *is the largest and the second largest in* $\{g_i, i \in N\}$.

(b) *For* $i \in N$, $X_i^* = f_i(X) - \mathbb{E}^{\mathbb{Q}}[f_i(X)] + \mathbb{E}^{\mathbb{Q}}[\xi_i]$ *almost surely where* $f_i$ *satisfies (2) with* $\sum_{i=1}^{n} f_i(X) = X$.

# Existence of CCE

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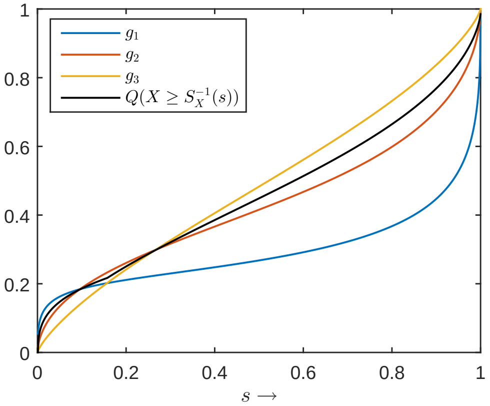
Sharp contrast I: A competitive equilibrium

- always exists in a DU-comonotone market;
- does NOT necessary exist in a DU-complete market where the distortion functions are not convex (e.g. Embrechts-Liu-Wang'18, the case of VaR)

# Example of equilibrium pricing measure




(a) Distortion functions and equilibrium price

# Uniqueness of CCE

In the DU-comonotone market,

(i) If $g_{N,1}(t) > g_{N,2}(t)$ for almost everywhere $t \in [0,1]$, then the equilibrium allocation is unique up to constant shifts, and the equilibrium price is not unique.

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(ii) If $g_{N,1}(t) = g_{N,2}(t)$ for almost everywhere $t \in [0,1]$, then the equilibrium price is unique, and the equilibrium allocation is not unique.

Sharp contrast II: The equilibrium price

- is unique in a DU-complete market; e.g. Boonen'15
- is NOT necessary unique in a DU-comonotone market.

# FTWE

## Theorem (2)

*In the DU-comonotone market,*

Without central coordination

# FTWE

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*In the DU-comonotone market,*

(i) *an equilibrium allocation is necessarily Pareto-optimal;*

Without central coordination

- 1st FTWE
  *"Invisible hand":* a competitive market leads to an efficient allocation of resources.

# FTWE

## Theorem (2)

*In the DU-comonotone market,*

(i) *an equilibrium allocation is necessarily Pareto-optimal;*

(ii) *a Pareto-optimal allocation is necessarily an equilibrium allocation for some choice of endowments.*

Without central coordination

- 1st FTWE
  *"Invisible hand":* a competitive market leads to an efficient allocation of resources.
- 2nd FTWE
  Any desired Pareto-efficient allocation can be attained by market competition with transfers.

Introduction

Competitive equilibria with dual utilities

Competitive equilibria with rank dependent utilities

An algorithm for computing competitive equilibria

Conclusion

# Competitive equilibria with rank-dependent utilities

In a comonotone market, where agents are equipped with rank-dependent utilities, we investigate following issues.

- Existence of a competitive equilibrium.
- First fundamental theorems of welfare economics.
- EU market approach.

# RDU-comonotone market

## Individual optimization:

$$\max_{X_i \in C(X)} V_i(X_i) = R_{u_i,g_i}(X_i) = \int_\Omega u_i(X_i)\mathrm{d}(g_i \circ \mathbb{P})$$
$$\text{s.t. } \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i]$$

or $X_i^* = Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i] + \mathbb{E}^{\mathbb{Q}}[\xi_i]$, where

$$Y_i \in \arg\max_{Y \in C(X)} \left\{ V_i \left( Y - \mathbb{E}^{\mathbb{Q}}[Y] + \mathbb{E}^{\mathbb{Q}}[\xi_i] \right) \right\}.$$

# RDU-comonotone market

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Recall: Constrained competitive equilibrium (CCE)

$((X_i^*, \ldots, X_n^*), \mathbb{Q}) \in \mathbb{A}_n^c \times \mathcal{P}$ is a CCE if $\mathbb{E}^{\mathbb{Q}}[X_i^*] = \mathbb{E}^{\mathbb{Q}}[\xi_i]$ and

$$V_i(X_i^*) = \max \left\{ V_i(Y_i) : Y_i \in C(X),\ \mathbb{E}^{\mathbb{Q}}[Y_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i] \right\}.$$

# Existence & FTWE

Assumptions. In a *RDU-comonotone market* or an *RDU-complete market* with given $\xi_1, \ldots, \xi_n, X \in \mathcal{X}$,

- $u_1, \ldots, u_n$ are strictly increasing, strictly concave and continuously differentiable functions and $u_i > -\infty$ on $(d_i, \infty)$, $i \in N$.
- $g_1, \ldots, g_n \in \mathcal{G}$ are continuously differentiable.

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- $g_1, \ldots, g_n \in \mathcal{G}$ are continuously differentiable.

## Theorem (3)

*Consider the RDU-comonotone market.*

1. *(Existence.) If $\xi_i \geq d_i$ and $\xi_i$ is a continuos function of $X$, $i \in N$, then a competitive equilibrium exists.*
2. *(1st FTWE.) An equilibrium allocation satisfying $V_i(X_i) > -\infty$ for all $i \in N$ is necessarily Pareto-optimal.*

# Expected-utility with heterogeneous beliefs

Observation.

$$V_i(Y) = R_{g_i,u_i}(Y) = \mathbb{E}^{Q_i}[u_i(Y)], \quad \text{for all} \quad Y \in C(X),$$

where $Q_i \in \mathcal{P}$, $i \in N$ such that $Q_i(X > t) = g_i \circ \mathbb{P}(X > t)$ for all $t \in \mathbb{R}$.

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In a comonotone market, the individual RDU optimization problem translates to a *EU problem with heterogeneous beliefs* $Q_i$, $i \in N$:

$$\max_{X_i \in C(X)} \mathbb{E}^{Q_i}[u_i(X_i)] \;\text{ s.t. }\; \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

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- $V_i(Y) = \mathbb{E}^{Q_i}[u_i(Y)]$ relies on the fact that $(Y, X)$ is comonotonic, and it does not necessarily hold on $\mathcal{X}$.

# RDU markets & EU markets

| | | |
|---|---|---|
| RDU-comonotone market<br>CCE | $\Longleftrightarrow$ | EU-comonotone market<br>CCE |
| $\Uparrow$ (generally) | | $\Uparrow$ (if comonotonic) |
| RDU-complete market<br>UCE | $\not\Longleftrightarrow$ | EU-complete market<br>UCE |
| | | (relatively well studied) |

Individual objectives:

- EU-comonotone market

$$\max_{X_i \in C(X)} \mathbb{E}^{Q_i}[u_i(X_i)] \;\text{ s.t. }\; \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

- EU-complete market

$$\max_{X_i \in \mathcal{X}} \mathbb{E}^{Q_i}[u_i(X_i)] \;\text{ s.t. }\; \mathbb{E}^{\mathbb{Q}}[X_i] \leq \mathbb{E}^{\mathbb{Q}}[\xi_i].$$

# Competitive equilibria in an EU-complete market

Optimization problems in the EU-complete market with heterogeneous beliefs $Q_i$, $i \in N$:

$$\max_{X_i \in \mathcal{X}} \mathbb{E}^{Q_i}[u_i(X_i)] \text{ s.t. } \mathbb{E}^{\mathbb{Q}}[X_i] \le \mathbb{E}^{\mathbb{Q}}[\xi_i], \quad i \in N.$$

- Individual optimization has a unique solution (e.g. Föllmer-Schied'16)

$$X_i = (u_i')^{-1}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}Q_i}\lambda_i\right), \quad \mathbb{E}^{\mathbb{Q}}[X_i] = \mathbb{E}^{\mathbb{Q}}[\xi_i]$$

- The market clearing condition

$$\sum_{i=1}^{n} (u_i')^{-1}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}Q_i}\lambda_i\right) = X,$$

## Theorem (4)

*Suppose that* $((X_1, \ldots, X_n), \mathbb{Q})$ *is an UCE in the EU-complete market. If*

$$\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}Q_1}, \ldots, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}Q_n}\right) \text{ is comonotonic ,}$$

*then it is a CCE in the RDU-comonotone market.*

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*then it is a CCE in the RDU-comonotone market.*

Sharp contrast III:

- In a CCE, the pricing kernel

$$\eta = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}Q_i} g_i'(S_X(X))$$

  is not necessarily a decreasing function of $X$ when $g_i$ is not convex.
- Our model could accommodate the *pricing kernel puzzle* that pricing kernel is not necessarily counter-monotonic with X by empirical observations (e.g. Hens-Reichlin'13).

# Exponential utilities

For $i \in N$, assume

$$u_i(x) = -e^{-\frac{x}{\theta_i}}, \quad x \in \mathbb{R},$$

where $\theta_1, \ldots, \theta_n > 0$ are parameters representing *risk tolerance*.

## Proposition

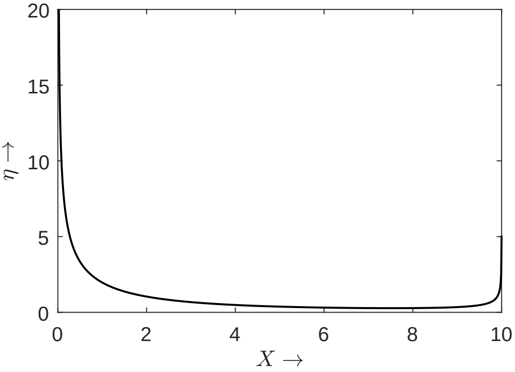
*With exponential utilities, if the following condition holds,*

$$\inf_{x\in\mathbb{R}} \inf_{j=1,\ldots,n} \left\{ \bar{\theta}^{-1} + \frac{q_j'(x)}{q_j(x)} - \sum_{i=1}^{n} \frac{\theta_i}{\bar{\theta}} \frac{q_i'(x)}{q_i(x)} \right\} \geq 0,$$

*where $\bar{\theta} = \sum_{i=1}^{n} \theta_i$ and $q_i(x) = \frac{\mathrm{d}Q_i(X \leq x)}{\mathrm{d}x}$, then a CCE is given by*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{ \frac{1}{\bar{\theta}} \left( \sum_{i=1}^{n} \theta_i \ln\left( \frac{\mathrm{d}Q_i}{\mathrm{d}\mathbb{P}} \right) + \bar{c} - X \right) \right\},$$

$$X_j = \frac{\theta_j}{\bar{\theta}} \left( X - \sum_{i=1}^{n} \theta_i \ln\left( \frac{\mathrm{d}Q_i}{\mathrm{d}Q_j} \right) - \bar{c} \right) + c_j.$$




(b) Equilibrium pricing kernal

# Idea of the algorithm

Discretization.

- Take $\hat{X} = \{x_1, \ldots, x_m\}$ such that $\varepsilon = x_{i+1} - x_i$ is small enough.
- The initial wealth of agent $i$ is $\psi_0^i = \sum_{k=1}^m \delta_{0,k}^i \mathbb{I}(x \geq x_k)$.
- The initial guess of the price is $q_{0,k} = \hat{Q}_0(\hat{X} \geq x_k)$, $k = 1, \ldots, m$.

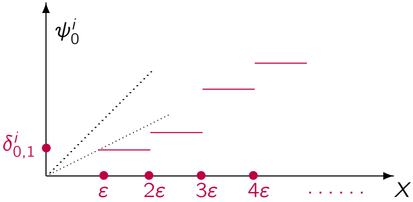
Initial input.

- $\hat{X}$, $\hat{Q}_0 = \hat{\mathbb{P}}$, $\psi_0^i = \hat{\xi}_i$ if $\xi_i \in C(X)$, otherwise $\psi_0^i = \frac{\mathbb{E}^{\hat{Q}_0}[\hat{\xi}_i]}{\mathbb{E}^{\hat{Q}_0}[\hat{X}]}\hat{X}$.

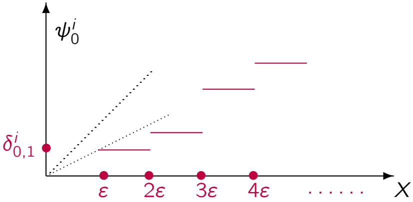
Updating process.

- In each step, we update $\delta_{0,k}^i \in [0, \varepsilon]$ and $q_{0,k}$ consequently such that each $\varepsilon$ is optimal allocated and the market is cleared.

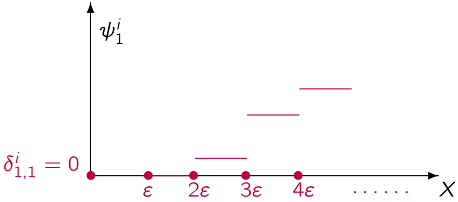
The initial wealth.

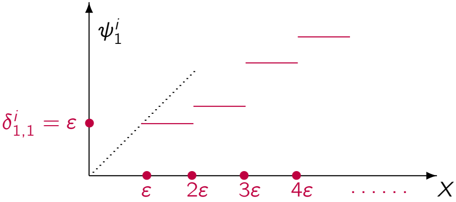


The initial wealth.




After the first step:



# Examples

Simple setup.

- $N = \{1, 2, 3\}$
- $X \sim U[0, 10]$, $\varepsilon = 0.01$, $m = 1000$
- $\xi_i = X/3$, $i = 1, 2, 3$

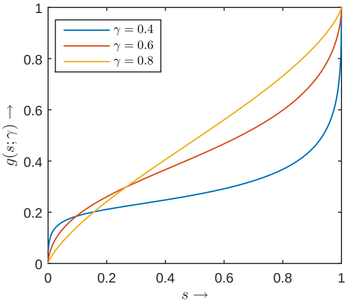
# Example 1 - Dual utility

Assumptions.

- Distortion functions, for $s \in [0, 1]$

$$g(s; \gamma_i) = \frac{s^{\gamma_i}}{(s^{\gamma_i} + (1 - s)^{\gamma_i})^{1/\gamma_i}},$$

where $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, and $\gamma_3 = 0.8$ (Tversky-Kahneman'92).




(c) Distortion functions

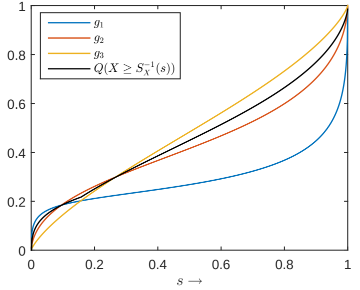
# Example 1 - Dual utility

- For the inverse-S shape distortion functions, UCE may not exist, but CCE exists.
- *Certainty equivalents (CEQ).* For $i \in N$, let $\mathrm{CEQ}_i^{\mathrm{prior}}$ and $\mathrm{CEQ}_i^{\mathrm{post}}$ be constants s.t.

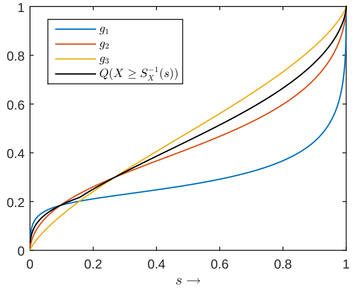
$$V_i(\mathrm{CEQ}_i^{\mathrm{prior}}) = V_i(\xi_i) \text{ and } V_i(\mathrm{CEQ}_i^{\mathrm{post}}) = V_i(X_i)$$

| | $\mathrm{CEQ}_i^{\mathrm{prior}}$ | $\mathrm{CEQ}_i^{\mathrm{post}}$ (theoretical/algorithm) | % increase |
|---|---|---|---|
| Agent 1 | 0.99 | 1.56 / 1.56 | 58.0 |
| Agent 2 | 1.44 | 1.56 / 1.56 | 8.3 |
| Agent 3 | 1.63 | 1.86 / 1.86 | 14.7 |

# Example 1 - Dual utility

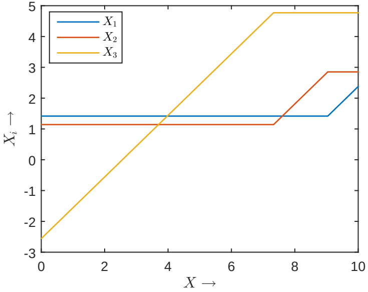



(d) Distortion functions and equilibrium price (exact)

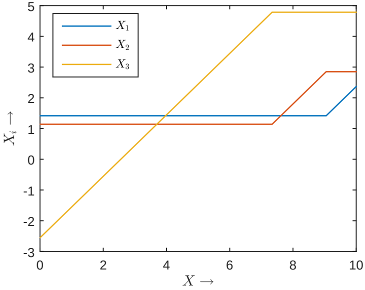



(e) Distortion functions and equilibrium price (algorithm)

# Example 1 - Dual utility




(f) Equilibrium allocation (exact)




(g) Equilibrium allocation (algorithm)

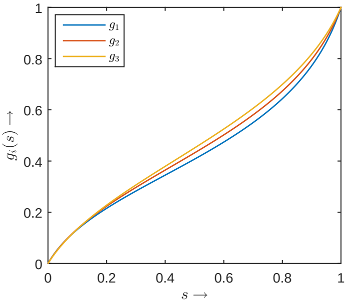
# Example 2 - RDU with explicit solutions

## Assumptions.

- Distortion functions, for $s \in [0,1]$

$$g_i(s) = ag\left(\frac{s+0.05}{1+2\delta}; \gamma_i\right) + b,$$

  where $\gamma_1 = 0.55$, $\gamma_2 = 0.6$, and $\gamma_3 = 0.65$.
- Exponential utilities $u_i(x) = -e^{-x/\theta_i}$ with $\theta_1 = 2$, $\theta_2 = 1.5$ and $\theta_3 = 1$.



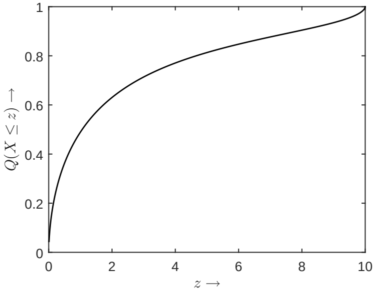

(h) Distortion functions

# Example 2 - RDU with explicit solutions

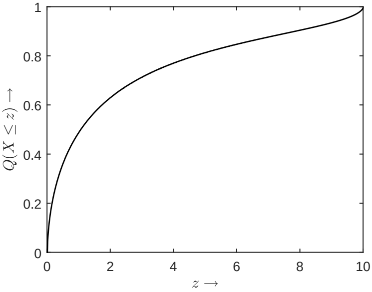
The certainty equivalents before and after risk sharing

| | $\mathrm{CEQ}_i^{\text{prior}}$ | $\mathrm{CEQ}_i^{\text{post}}$ (theoretical/algorithm) | % increase |
|---|---|---|---|
| Agent 1 | 1.156 | 1.167 / 1.167 | 0.9 |
| Agent 2 | 1.138 | 1.138 / 1.138 | 0 |
| Agent 3 | 1.049 | 1.070 / 1.069 | 2.0 |

# Example 2 - RDU with explicit solutions

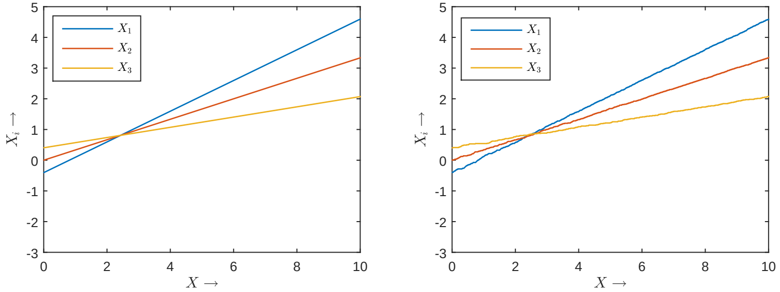



(i) Equilibrium price (exact)




(j) Equilibrium price (algorithm)

# Example 2 - RDU with explicit solutions




(k) Equilibrium allocation (exact) (l) Equilibrium allocation (algorithm)

# Example 3 - RDU without explicit solutions

Assumptions.

- The three agents use distortion functions

$$g(s;\gamma_i) = \frac{s^{\gamma_i}}{(s^{\gamma_i} + (1-s)^{\gamma_i})^{1/\gamma_i}}, \quad s \in [0,1],$$

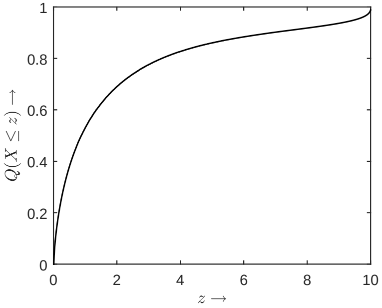
  where $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, and $\gamma_3 = 0.8$.
- Exponential utility $u_i(x) = -e^{-x/\theta_i}$ with risk tolerant $\theta_1 = 3$, $\theta_2 = 2$ and $\theta_3 = 1$.

# Example 3 - RDU without explicit solutions

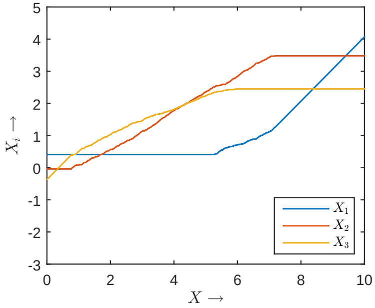
- The equilibrium is most attractive for the agent with the most distorted probability measure (Agent 1) and for the most risk averse agent (Agent 3).

| | $\mathrm{CEQ}_i^{\text{prior}}$ | $\mathrm{CEQ}_i^{\text{post}}$ (algorithm) | % increase |
|---|---|---|---|
| Agent 1 | 0.75 | 0.90 | 19.3 |
| Agent 2 | 1.10 | 1.14 | 3.0 |
| Agent 3 | 1.11 | 1.19 | 6.8 |

# Example 3 - RDU without explicit solutions




(m) Equilibrium price (algorithm)




(n) Equilibrium allocation (algorithm)

# Summary of our work

- Introducing the comonotome market.
- Solving competitive equilibrium problem in a DU-comonotone market.
  - Existence and closed form.
  - Fundamental theorems of welfare economics.
- Partially solving competitive equilibrium problem in a RDU-comonotone market by a EU approach.
  - Existence.
  - Obtaining CCE under exponential utilities (and power utilities).
- Proposing an algorithm on determining CCE in general cases.

# Open questions in RDU-comonotone market

- Existence of competitive equilibrium under more general assumptions.
- Uniqueness of competitive equilibrium.
- The second fundamental theorem of welfare economics for the RDU market.
- If $Q_1 = \cdots = Q_n$, that is a EU market with the same belief. Then a CCE in EU market is also a CCE in RDU market. Is it the only equilibrium for the comonotone market?
- Whether the EU-comonotone market and the EU-complete market have the same equilibria?
- More ...

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# Thank you

Thank you for your kind attention