Sum of Two Standard Uniform Random Variables

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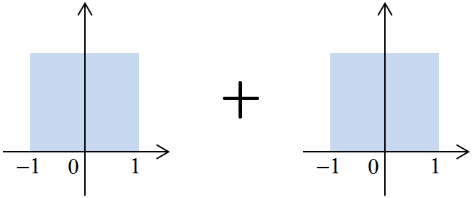
A Question

In this talk we discuss this problem:

$$X_1 \sim \mathrm{U}[-1,1],\ X_2 \sim \mathrm{U}[-1,1]$$

what is a distribution (cdf) of $X_1 + X_2$?



A difficult problem with no applications (?)

Generic Formulation

In an atomless probability space:

- $F_1, \ldots, F_n$ are n distributions
- $X_i \sim F_i$, $i = 1, \ldots, n$
- $S_n = X_1 + \cdots + X_n$

Denote the set of possible aggregate distributions

$$\mathcal{D}_n = \mathcal{D}_n(F_1, \cdots, F_n) = \{\text{cdf of } S_n | X_i \sim F_i,\ i = 1, \cdots, n\}.$$

Primary question: Characterization of $\mathcal{D}_n$.

- $\mathcal{D}_n$ is non-empty, convex, and closed w.r.t. weak convergence

Generic Formulation

For example:

- X_i: individual risks; S_n: risk aggregation
- fixed marginal; unknown copula

Classic setup in Quantitative Risk Management

- Secondary question: what is $\sup_{F \in \mathcal{D}_n} \rho(F)$ for some functional ρ (risk measure, utility, moments, ...)?
- Risk aggregation with dependence uncertainty, an active field over the past few years:
  - Embrechts et. al. (2014 Risks) and the references therein
  - Books: Rüschendorf (2013), McNeil-Frey-Embrechts (2015)
  - 20+ papers in the past 3 years

Some Observations

Assume that $F_1, \dots, F_n$ have finite means $\mu_1, \dots, \mu_n$, respectively.

- Necessary conditions:
  - $S_n \prec_{\mathrm{cx}} F_1^{-1}(U) + \cdots + F_n^{-1}(U)$
    - In particular, $\mathbb{E}[S_n] = \mu_1 + \cdots + \mu_n$
  - $\mathrm{Range}(S_n) \subset \sum_{i=1}^n \mathrm{Range}(X_i)$
- Suppose $\mathbb{E}[T] = \mu_1 + \cdots + \mu_n$. Then

$$F_T \in \mathcal{D}_n(F_1, \dots, F_n) \Leftrightarrow (F_{-T}, F_1, \dots, F_n) \text{ is jointly mixable}$$

For a theory of joint mixability

- W.-Peng-Yang (2013 FS), Wang-W. (2016 MOR)
- Surveys: Puccetti-W. (2015 STS), W. (2015 PS)
- Numerical method: Puccetti-W. (2015 JCAM)

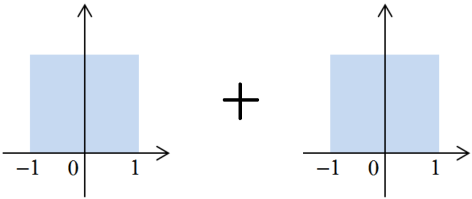
Some Observations

- Joint mixability is an open research area
- A general analytical characterization of $\mathcal{D}_n$ or joint mixability is far away from being clear
- We tune down and look at standard uniform distributions and $n = 2$

Progress of the Talk

Simple Examples

$X_1 \sim \mathrm{U}[-1,1]$, $X_2 \sim \mathrm{U}[-1,1]$, $S_2 = X_1 + X_2$.



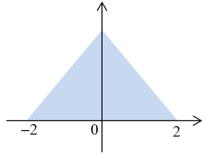
Obvious constraints

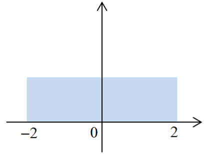
- $\mathbb{E}[S_2] = 0$
- range of S_2 in $[-2, 2]$
- $\mathrm{Var}(S_2) \le 4/3$

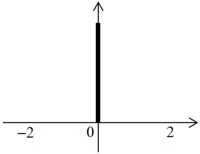
- $S_2 \prec_{\mathrm{cx}} 2X_1$ (sufficient?)

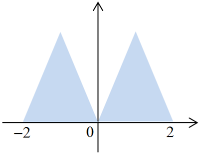
Simple Examples

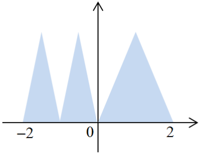
Are the following distributions possible for S_2?

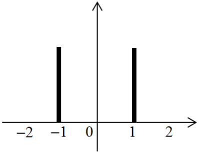




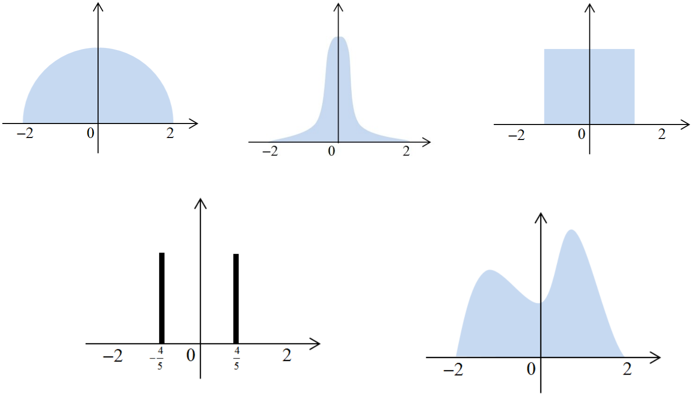








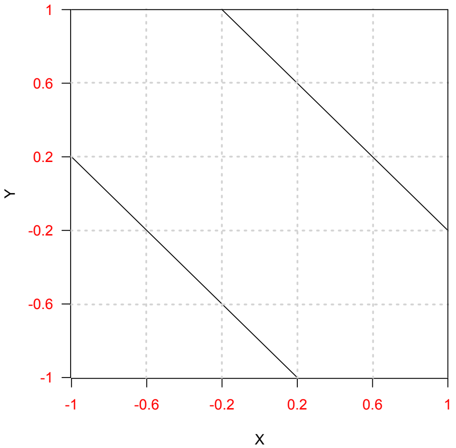
Simple Examples: More...



Examples and counter-examples: Mao-W. (2015 JMVA) and Wang-W. (2016 MOR)

A Small Copula Game...

$$\mathbb{P}(S_2 = -4/5) = 1/2, \ \mathbb{P}(S_2 = 4/5) = 1/2$$



Progress of the Talk

Existing Results

Let $\mathcal{D}_2 = \mathcal{D}_2(\mathrm{U}[-1,1], \mathrm{U}[-1,1])$. Below are implied by results in Wang-W. (2016 MOR)

- Let F be any distribution with a monotone density function. then $F \in \mathcal{D}_2$ if and only if F is supported in $[-2,2]$ and has zero mean.
- Let F be any distribution with a unimodal and symmetric density function. Then $F \in \mathcal{D}_2$ if and only if F is supported in $[-2,2]$ and has zero mean.
- $\mathrm{U}[-a,a] \in \mathcal{D}_2$ if and only if $a \in [0,2]$ (a special case of both).
  - The case $\mathrm{U}[-1,1] \in \mathcal{D}_2$ is given in Rüschendorf (1982 JAP).

Unimodal Densities

A natural candidate to investigate is the class of distributions with a unimodal density.

Theorem 1

Let F be a distribution with a unimodal density on $[-2, 2]$ and zero mean. Then $F \in \mathcal{D}_2$.

- Both the two previous results are special cases
- For bimodal densities we do not have anything concrete

Densities Dominating a Uniform

A second candidate is a distribution which dominates a portion of a uniform distribution.

Theorem 2

Let F be a distribution supported in $[a-b, a]$ with zero mean and density function f. If there exists $h > 0$ such that $f \geq \frac{3b}{4h}$ on $[-h/2, h/2]$, then $F \in \mathcal{D}_2$.

- The density of F dominates $3b/4$ times that of $\mathrm{U}[-h/2, h/2]$

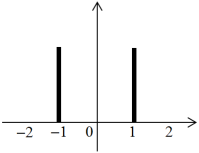
Bi-atomic Distributions

Continuous distributions seem to be a dead end; what about discrete distributions? Let us start with the simplest cases.

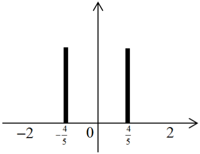
Bi-atomic Distributions

Theorem 3

Let F be a bi-atomic distribution with zero mean supported on $\{a-b, a\}$. Then $F \in \mathcal{D}_2$ if and only if $2/b \in \mathbb{N}$.



$2/b = 1$



$2/b = 5/4$

- For given $b > a > 0$, there is only one distribution on $\{a-b, a\}$ with mean zero.

Tri-atomic Distributions

For a tri-atomic distribution F, write $F = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are the probability masses of F

- On given three points, the set of tri-atomic distributions with mean zero has one degree of freedom.
- We study the case of F having an "equidistant support" $\{a-2b, a-b, a\}$.

For $x > 0$, define a "measure of non-integrity"

$$\lceil x \rfloor = \min\left\{\frac{\lceil x \rceil}{x} - 1, 1 - \frac{\lfloor x \rfloor}{x}\right\} \in [0,1].$$

Obviously $\lceil x \rfloor = 0 \Leftrightarrow x \in \mathbb{N}$.

Tri-atomic Distributions

Theorem 4

Suppose that $F = (f_1, f_2, f_3)$ is a tri-atomic distribution with zero mean supported in $\{a-2b, a-b, a\}$, $\epsilon > 0$ and $a \leq b$. Then $F \in \mathcal{D}_2$ if and only if it is the following three cases.

(i) $a = b$ and $f_2 \geq \lceil \frac{1}{b} \rfloor$.

(ii) $a < b$ and $\frac{1}{b} \in \mathbb{N}$.

(iii) $a < b$, $\frac{1}{b} - \frac{1}{2} \in \mathbb{N}$ and $f_2 \geq \frac{a}{2}$.

- cf. Theorem 3 (condition $2/b \in \mathbb{N}$)

Tri-atomic Distributions

The corresponding distributions in Theorem 4:

(i) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0,1,0), \frac{1}{2}(1 - \lceil \frac{1}{b} \rfloor, 2\lceil \frac{1}{b} \rfloor, 1 - \lceil \frac{1}{b} \rfloor)\}$.

(ii) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0, \frac{a}{b}, 1 - \frac{a}{b}), \frac{1}{2}(\frac{a}{b}, 0, 2 - \frac{a}{b})\}$.

(iii) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0, \frac{a}{b}, 1 - \frac{a}{b}), \frac{1}{2}(\frac{a}{b} - \frac{a}{2}, a, 2 - \frac{a}{b} - \frac{a}{2})\}$.

Progress of the Talk

Some More to Expect

- It is possible to further characterize n-atomic distributions with an equidistant support (things get ugly though).
- We guess: for any distribution F
  - with an equidistant support, or
  - with finite density and a bounded support,

  there exists a number $M > 0$ such that
  $F \in \mathcal{D}_2(\mathrm{U}[-m, m], \mathrm{U}[-m, m])$ for all $m \in \mathbb{N}$ and $m > M$.

Some More to Think

- Two uniforms with different lengths?
- Three or more uniform distributions?
- Other types of distributions?
- Applications?

We yet know very little about the problem of $\mathcal{D}_2$

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Danke Schön

Thank you for your kind attention