Risk Measurement: History, Trends and Challenges

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PKU-Math International Workshop on Financial Mathematics
Peking University Beijing, China August 18, 2014

Outline

WATRISQ

NSERC CRSNG

Introduction

Key question in mind

A financial institution has a risk (random loss) X in a fixed period. How much capital should this financial institution reserve in order to undertake this risk?

- X can be financial risks, credit risks, operational risks, insurance risks, etc.
- Regulator's viewpoint
- Risk manager's viewpoint

Risk Measures

First, a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- $\mathbb{P}$-a.s. equal random variables are treated as identical.

A risk measure is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

- $\mathcal{X} \supset L^\infty$ is a set which is closed under addition and $\mathbb{R}^+$-multiplication.
- Typically one requires $\rho(L^\infty) \subset \mathbb{R}$ for obvious reasons.

Example: VaR

$p \in (0,1)$, $X \sim F$.

Definition 1 (Value-at-Risk)

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}.$$

Example: ES

$p \in (0,1)$.

Definition 2 (Expected Shortfall (TVaR, CVaR, CTE, WCE))

$\mathrm{ES}_p : L^0 \to (-\infty, \infty]$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F \text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_p(X)\right].$$

In addition, let $\mathrm{VaR}_1(X) = \mathrm{ES}_1(X) = \text{ess-sup}(X)$, and $\mathrm{ES}_0(X) = \mathbb{E}[X]$ (only well-defined on e.g. L^1 or L^0_+).

Example: Standard Deviation Principle

$b \geq 0$.

Definition 3 (Standard deviation principle)

$\mathrm{SD}_b : L^2 \to \mathbb{R}$,

$$\mathrm{SD}_b(X) = \mathbb{E}[X] + b\sqrt{\mathrm{Var}(X)}.$$

- A small note: for normal risks, one can find p, q, b such that $\mathrm{VaR}_p(X) = \mathrm{ES}_q(X) = \mathrm{SD}_b(X)$. Example: $p = 0.99$, $q = 0.975$, $b = 2.33$.

Functionals: $\mathcal{X} \to [-\infty, \infty]$

Three major perspectives

- Preference of risk: Economic Decision Theory
- Pricing of risk: Insurance and Actuarial Science
- Capital requirement: Mathematical Finance

Preference of Risk

Preference of risk: Economic Decision Theory

- Mathematical theory established since 1940s.
  - Expected utility: von Neumann and Morgenstern (1944).
  - Rank-dependent utility: Quiggin (1982, JEBO).
  - Dual utility: Yaari (1987, Econometrica); Schmeidler (1989, Econometrica).

Pricing of Risk

Pricing of risk: Insurance and Actuarial Science

- Mathematical theory established since 1970s.
  - Additive principles: Geber (1974, ASTIN Bulletin).
  - Economic principles: Bühlmann (1980, ASTIN Bulletin).
  - Convex principles: Deprez and Gerber (1985, IME).
  - Axiomatic principles: Wang, Young and Panjer (1997, IME).

Capital Requirement

Capital requirement: Mathematical Finance

- Mathematical theory established around 1999.
  - Coherent measures of risk: Artzner, Delbaen, Eber and Heath (1999, MF).
    - Citation: 5500+ (Google, Aug 2014)
  - Law-invariant risk measures: Kusuoka (2001, AME).
  - Convex measures of risk: Föllmer and Schied (2002, FS).
  - Spectral measures of risk: Acerbi (2002, JBF).
- Mathematically very well developed, and fast expanding in the past 15 years.
- Value-at-Risk introduced earlier (around 1994): e.g. Duffie and Pan (1997, J. Derivatives).

Caution...

Different perspectives should lead to different principles of desirability.

- Preference of risk: only ordering matters (not precise values), gain and loss matter
- Pricing of risk: precise values matter, gain and loss matter
  - central limit theorem often kicks in (large number effect)
  - typically there is a market
- Capital requirement: precise values matter, only loss matters (← our focus)
  - typically there is no market; no large number effect

Of course, mathematically very much overlapping...

Research of Risk Measures

Two major perspectives

- What interesting mathematical/statistical problems arise from this field?
- What risk measures are practical in real life, and what are the practicality issues?

Good research may address both questions, but it often only addresses one of them.

Monetary Risk Measures

Two basic properties

- cash-invariance: $\rho(X+c) = \rho(X) + c$, $c \in \mathbb{R}$;
- monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

(A monetary risk measure)

- Financial interpretations of the above properties are clear.
- Here, risk-free interest rate is assumed to be 0 (everything is discounted).
- In particular: $\rho(X - \rho(X)) = 0$.

Monetary Risk Measures

- VaR_p, $p \in (0,1)$ is monetary;
- ES_p, $p \in (0,1)$ is monetary;
- SD_b, $b > 0$ is cash-invariant, but not monotone.

Acceptance Sets

The acceptance set of a risk measure ρ:

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}.$$

- Example: $\mathcal{A}_{\mathrm{VaR}_p} = \{X \in L^0 : \mathbb{P}(X \leq 0) \geq p\}$.
- Financial interpretation: the set of risks that are considered acceptable by a regulator or manager.
- A cash-invariant risk measure ρ is fully characterized by its acceptance set.

Acceptance Sets

Theorem: Duality

Let $\mathcal{A}$ be any lower-subset of $\mathcal{X}$ containing at least a constant. Then

$$\rho_{\mathcal{A}}(X) = \inf\{m : X - m \in \mathcal{A}\}$$

is a monetary risk measure. Moreover, for any monetary risk measure ρ,

$$\rho(X) = \rho_{\mathcal{A}_\rho}(X).$$

- First version established in ADEH (1999).
- Financial interpretation: $\rho_{\mathcal{A}}(X)$ is the amount of money required to make X acceptable.

Relation to Finance

Instead of a zero-interest bond, one may think about a general security S with $S_0 = 1$.

A risk measure can be defined as

$$\rho_{\mathcal{A}}(X) = \inf\{m : X - mS_T \in \mathcal{A}\}.$$

Relation to Finance

We may have multiple securities in a financial market.

A risk measure can be defined as

$$\rho_{\mathcal{A}}(X) = \inf\{m : X - \pi_T \in \mathcal{A},\ \pi \in \Pi,\ \pi_0 = m\}.$$

where Π is the set of admissible self-financing portfolios.

- Example: $\mathcal{A} = \{X \in \mathcal{X} : X \leq 0\ \ \mathbb{P}\text{-a.s.}\}$.
    - This means the regulator only accepts profit, not any loss.
    - $\rho_{\mathcal{A}}(X)$ is the superhedging price of X.
    - In a complete market, it is the arbitrage-free price of X.
    - If only a zero-interest bond is available (original setting), then $\rho_{\mathcal{A}}(X) = \text{ess-sup}(X)$.

Coherent and Convex Risk Measures

Two more properties in addition to being monetary

- positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \in \mathbb{R}^+$;
- subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

(A coherent risk measure; ADEH, 1999)

- subadditivity can be replaced by convexity:
  $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\lambda \in [0, 1]$.

(A monetary risk measure that is convex, is called a convex risk measure; Föllmer and Schied, 2002)

One can easily check that ES is coherent but VaR is not; the latter is not subadditive (or convex).

Subadditivity

Subadditivity arguments:

- diversification benefit - *"a merger does not create extra risk"*;
- regulatory arbitrage: divide X into $Y + Z$ if $\rho(X) > \rho(Y) + \rho(Z)$;
- capturing the tail risk;
- consistency with risk preference;
- convex optimization and capital allocation.

Subadditivity

Subadditivity is contested from different perspectives:

- aggregation penalty - *convex risk measures*;
- statistical inference - *estimation/robustness/elicitability*;
- financial practice - *"a merger creates extra risk"*;
- legal consideration - *"an institution has limited liability"*.

Coherent and Convex Risk Measures

Theorem: ADEH, 1999

A monetary risk measure is coherent if and only if its acceptance set is a convex cone.

Theorem: Föllmer and Schied, 2002

A monetary risk measure is convex if and only if its acceptance set is convex.

Examples of Convex Risk Measures

Shortfall risk measures:

$$\rho(X) = \inf\{y \in \mathbb{R} : \mathbb{E}[\ell(X - y)] \le \ell(0)\}.$$

ℓ: convex and increasing function.

- Motivated from indifference pricing: the acceptance set of ρ is
$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \mathbb{E}[\ell(X)] \le \ell(0)\}.$$
- Example: $\ell(x) = e^{tx}$, $t > 0$, then $\rho(X) = \frac{1}{t} \log \mathbb{E}[e^{tX}]$, the entropic risk measure.
- Example: $\ell(x) = px_+ - (1-p)x_-$, $p \in [1/2, 1)$, then $\rho(X)$ is the p-expectile (see Bellini, Klar, Müller and Rosazza Gianin, 2014, IME).

Main Theorem

Now suppose Ω is a finite set and $\mathcal{X}$ consists of all random variables in this probability space.

Theorem: ADEH, 1999; Huber, 1980.

A coherent risk measure ρ has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X}$$

where $\mathcal{R}$ is a collection of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

Expected Shortfall

Representation of Expected Shortfall

For $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \; X \in \mathcal{X},$$

where $\mathcal{R} = \{Q \text{ is a probability measure} : \mathrm{d}Q/\mathrm{d}\mathbb{P} \leq 1/(1-p)\}$.

Main Theorem

Now suppose Ω is general and $\mathcal{X} = L^\infty$ (throughout the rest of this talk).

Theorem: Delbaen, 2000

A coherent risk measure ρ has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X}$$

where $\mathcal{R}$ is a subset of **Ba** with $Q(\Omega) = 1$, $Q \in \mathcal{R}$, and **Ba** is the dual space of L^∞.

- **Ba** is the set of bounded finitely additive measures absolutely continuous w.r.t. $\mathbb{P}$. **Ba** $\supset L^1$.

Continuity of Risk Measures

Fatou property

Fatou property: suppose $X, X_1, X_2, \dots \in \mathcal{X} = L^\infty$, $\sup_{k\in\mathbb{N}} ||X_k||_\infty < \infty$ and $X_k \to X$ a.s., then

$$\liminf_{k\to\infty} \rho(X_k) \geq \rho(X).$$

- Fatou property
  $\Leftrightarrow$ ρ is continuous from below (a.s. or $\mathbb{P}$ convergence)
  $\Leftrightarrow$ $\mathcal{A}_\rho$ is closed under the weak* topology $\sigma(L^\infty, L^1)$.

Remark

There is no coherent/convex risk measure ρ that is continuous w.r.t. a.s. convergence in L^∞.

Main Theorem

More results from Functional Analysis...

Theorem: Delbaen, 2000

A coherent risk measure ρ with the Fatou property has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \; X \in \mathcal{X}$$

where $\mathcal{R}$ is a collection of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

Law-invariant Coherent Risk Measures

One more important property from a statistical viewpoint...

- law-invariance: $\rho(X) = \rho(Y)$ if $X \stackrel{d}{=} Y$.

Theorem: Kusuoka, 2001

A law-invariant coherent risk measure with the Fatou property has the following representation:

$$\rho(X) = \sup_{h \in \mathcal{Q}_I} \int_0^1 \mathrm{ES}_p(X) \mathrm{d}h(p), \ X \in \mathcal{X}$$

where $\mathcal{Q}_I$ is a collection of probability measures on $[0, 1]$.

Law-invariant Coherent Risk Measures

One more important property from an economic viewpoint...

- comonotonic additivity: $\rho(X+Y)=\rho(X)+\rho(Y)$ if X and Y are comonotonic.

Theorem: Kusuoka, 2001; Yaari, 1987

A law-invariant and comonotonic additive coherent risk measure has the following representation:

$$\rho(X)=\int_0^1 \mathrm{ES}_p(X)\mathrm{d}h(p),\ X\in\mathcal{X}$$

where h is a probability measure on $[0,1]$.

Distortion Risk Measures

Theorem: Wang, Young and Panjer, 1997; Yaari, 1987

A law-invariant and comonotonic additive monetary risk measure has the following representation:

$$\rho(X) = \int_{\mathbb{R}} x \mathrm{d}h(F(x)), \; X \in \mathcal{X}, \; X \sim F$$

where h is a probability measure on $[0, 1]$.

ρ is called a distortion risk measure (DRM). h: its distortion function.

- ES and VaR are special cases of distortion risk measures.

Convex Risk Measures

Theorem: Föllmer and Schied, 2002; Frittelli and Rosazza Gianin, 2002, JBF

A convex risk measure ρ with the Fatou property has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{\mathbb{E}^Q[X] - a(Q)\}, \ X \in \mathcal{X}$$

where $\mathcal{P}$ is the set of probability measures absolutely continuous w.r.t. $\mathbb{P}$, and $a : \mathcal{P} \to (-\infty, \infty]$ is called a penalty function.

Convex Risk Measures

Theorem: Frittelli and Rosazza Gianin, 2005, AME

A law-invariant convex risk measure with the Fatou property has the following representation

$$\rho(X) = \sup_{h \in \mathcal{P}_I} \left\{ \int \mathrm{ES}_p(X) \mathrm{d}h(p) - a(h) \right\}, \; X \in \mathcal{X}$$

where $\mathcal{P}_I$ is the set of probability measures on $[0, 1]$, and $a : \mathcal{P}_I \to (-\infty, \infty]$ is a penalty function.

Convex Order

Convex order: $X \leq_{\text{cx}} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions f such that the expectations exist.

Theorem: Bäuerle and Müller, 2006, IME

A law-invariant convex risk measure with the Fatou property preserves convex order.

Convex Order

Finally, some of my own work:

Theorem: W. and Mao (2014, Working paper)

A monetary risk measure ρ preserves convex order if and only if it has the following representation:

$$\rho(X) = \inf_{\tau \in \mathcal{C}} \tau(X)$$

where $\mathcal{C}$ is a collection of law-invariant convex risk measure with the Fatou property.

More Results

- Extension to L^q, $q \in [1, \infty)$: see e.g. Kaina and Rüschendorf (2009, MMOR) and Filipović and Svindland (2012, MF).
- More mathematical results are available in the two major books: Delbaen (2012) and Föllmer and Schied (2011); I cannot exhaust them here.

New Trends

Situation:

VaR has been dominating in industry for the past decade.

Many academics (mainly mathematicians) advocate ES for it is coherent.

Basel Documents

From the Basel Committee on Banking Supervision:

R1: Consultative Document, May 2012,
Fundamental review of the trading book

R2: Consultative Document, October 2013,
Fundamental review of the trading book: A revised market risk framework.

Basel Question

R1, Page 41, Question 8:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

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- ES is not robust, whereas VaR is.
- The backtesting of ES is difficult, whereas that of VaR is straightforward.

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- The backtesting of ES is difficult, whereas that of VaR is straightforward.

Review paper: Embrechts, Puccetti, Rüschendorf, W. and Beleraj (2014, Risks).

Robustness

(Huber-Hampel's) robustness (see Huber and Ronchetti, 2007) usually refers to the continuity of a statistical functional $\rho : \mathcal{D} \to \mathbb{R}$ where $\mathcal{D}$ is a set of distribution functions.

- The strongest sense of continuity is w.r.t. weak topology.
- VaR_p is continuous if and only if $\mathcal{D}$ is chosen as the set of distributions that is absolutely continuous at its p-th quantile.
- ES_p is not continuous w.r.t. weak topology. It is continuous w.r.t. some stronger metric, e.g. the Wasserstein metric; see Stahl, Zheng, Kiesel and Rühlicke (2012).

Robustness

Robustness - some quotes

- Cont, Deguest and Scandolo (2010): "Our results illustrate in particular, that using recently proposed risk measures such as CVaR/Expected Shortfall leads to a less robust risk measurement procedure than Value-at-Risk."
- Kou, Peng and Heyde (2013, MOR): "Coherent risk measures are not robust".
- Emmer, Kratz and Tasche (2014): "The fact that VaR does not cover tail risks 'beyond' VaR is a more serious deficiency although ironically it makes VaR a risk measure that is more robust than the other risk measures we have considered."

Robustness

Example: different internal models

- Same data set, two different parametric models (e.g. normal vs student-t).
- Estimation of parameters, and compare the VaR and ES for two models.
- VaR is more robust in this setting, since it does not take the tail behavior into account (normal and student-t do not make a big difference).
- ES is less robust (heavy reliance on the model's tail behavior).
- Capital requirements: heavily depends on the internal models.

Robustness

Opposite opinions

- Cambou and Filipovic (2014): "In contrast to value-at-risk, expected shortfall is always robust with respect to minimum L^p-divergence modifications of $\mathbb{P}$."
- Krätschmer, Schied and Zähle (2014, FS): "Hampel's classical notion of qualitative robustness is not suitable for risk measurement ..." (introduced an index of qualitative robustness; ES has an index of 1 which is the best-possible index over all convex risk measures).

Robustness

Opposite opinions

- BCBS (2013, R4): "This confidence level [97.5th ES] will provide a broadly similar level of risk capture as the existing 99th percentile VaR threshold, while providing a number of benefits, including generally more stable model output and often less sensitivity to extreme outlier observations."
- Embrechts, Wang and W. (2014): "coherent distortion risk measures, including ES, are aggregation-robust while VaR is not." Also showed that VaR_p has a larger dependence-uncertainty spread compared to ES_q, $q \le p$.

Backtesting

Backtesting:

(i) estimate a risk measure from past observations;

(ii) test whether (i) is appropriate using future observations;

(iii) purpose: monitor, test or update risk measure forecasts.

Backtesting

Example - VaR backtesting:

(1) suppose the estimated/modeled VaR is V at $t = 0$;

(2) consider $A_t = \mathrm{I}_{\{X_t > V\}}$ based new iid observations X_t, $t > 0$;

(3) standard hypothesis testing methods for H_0: A_t are iid Bernoulli$(1 - \alpha)$ random variables.

For ES such simple and intuitive backtesting techniques do not exist!

Backtesting

Elicitability

- A new notion for comparing risk measure forecasts: elicitability; Gneiting (2011).
- Roughly speaking, a risk measure (statistical functional) $\rho : \mathcal{P} \to \mathbb{R}$ is elicitable if ρ is the unique solution to the following equation:

$$\rho(L) = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \mathbb{E}[s(x, L)],$$

where

  - $s : \mathbb{R}^2 \to [0, \infty)$ is a strictly consistent scoring function;
  - for example, the mean is elicitable with $s = (x - L)^2$.

Perspective of a Risk Analyst

Elicitability and comparison

- The estimated/modeled value of ρ is ρ_0 at $t = 0$;
- based on new iid observations X_t, $t > 0$, consider the statistics $s(\rho_0, X_t)$; for instance, test statistic can typically be chosen as $T_n(\rho_0) = \frac{1}{n}\sum_{t=1}^{n} s(\rho_0, X_t)$;
- $T_n(\rho_0)$: a statistic which indicates the goodness of forecasts.
- updating ρ: look at a minimizer for $T_n(\rho)$;
- the above procedure is model-independent.

Elicitable statistics are straightforward to backtest.

Perspective of a Regulator

Elicitability and regulation

- A value of risk measure ρ_0 is reported by a financial institution based on internal models.
- A regulator does not have access to the internal model, and she does not know whether ρ_0 is calculated honestly.
- She applies $s(\rho_0, X_t)$ as a daily penalty function for the financial insitution.
- If the institution likes to minimize this penalty, it has to report the true value of ρ and use the most realistic model.
- the above procedure is model-independent.

Elicitability

VaR vs ES: elicitability

Theorem: Gneiting, 2011, JASA

Under general conditions,

- VaR is elicitable;
- ES is not elicitable.

Elicitability

Remarks:

- under specific EVT-based conditions, backtesting of ES is possible; see McNeil, Frey and Embrechts (2005);
- the relevance of elicitability for risk management purposes is heavily contested:
    - Emmer, Kratz and Tasche (2014): alternative method for backtesting ES; favors ES.
    - Davis (2014): backtesting based on prequential principle; favors quantile-based statistics (VaR-type).

Elicitable Risk Measures

The following hold:

- if ρ is coherent, comonotonic additive and elicitable, then ρ is the mean (Ziegel, 2014, MF);
- if ρ is coherent and elicitable with a convex scoring function, then ρ is an expectile (Bellini and Bignozzi, 2014, QF);
- if ρ is comonotonic additive and elicitable, then ρ is a VaR or the mean (Kou and Peng, 2014).

Risk Aggregation and Splitting

Question: given a non-subadditive risk measure,

How superadditive can it be?

Risk Aggregation and Splitting

Question: given a non-subadditive risk measure,

How superadditive can it be?

Motivation:

- Measure model uncertainty
- Quantify worst-scenarios
- Trade subadditivity for statistical advantages
- Understand better about subadditivity

Two Perspectives

$$\rho\left(\sum_{i=1}^{n} X_i\right) \quad \text{against} \quad \sum_{i=1}^{n} \rho(X_i)$$

- Aggregation: fixed $X_i \sim F_i$, what is the worst-case aggregate value if arbitrary dependence is allowed in a portfolio?
- Division: fixed $X = \sum_{i=1}^{n} X_i$, what is the best-case aggregate value if arbitrary division is allowed in a position?

Diversification Ratio

For a law-invariant risk measure ρ, and risks $\mathbf{X} = (X_1, \cdots, X_n)$, the diversification ratio is defined as

$$\Delta^{\mathbf{X}}(\rho) = \frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)}.$$

For the moment, the denominator is assumed to be positive.

Diversification Ratio

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For the moment, the denominator is assumed to be positive.

- $\Delta^{\mathbf{X}}(\rho)$ is important in modeling portfolios.
- $\Delta^{\mathbf{X}}(\rho) \leq 1$ for subadditive risk measures.

Diversification Ratio

Fix F, define

$$\Delta_n^F(\rho) = \sup \left\{ \frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)} : X_1, \ldots, X_n \sim F \right\}.$$

Here we assumed homogeneity in F_i:

- mathematical tractability;
- to let n vary;
- $\Delta_n^{(\cdot)}(\rho) : \mathcal{D} \to \mathbb{R}$.

Question: $\Delta_n^F(\rho) \approx 1$?

Diversification Ratio

Define

$$\mathfrak{S}_n(F) = \{X_1 + \cdots + X_n : X_1, \ldots, X_n \sim F\}.$$

Let $X_F \sim F$. Then

$$\Delta_n^F(\rho) = \frac{1}{n\rho(X_F)} \sup \{\rho(S) : S \in \mathfrak{S}_n(F)\}.$$

- A challenging problem: W., Peng and Yang (2013, FS); Embrechts, Puccetti and Rüschendorf (2013, JBF).

Extreme-aggregation Measure

We are interested in the global superadditivity ratio

$$\Delta^F(\rho) = \sup_{n\in\mathbb{N}} \Delta_n^F(\rho) = \sup_{n\in\mathbb{N}} \frac{1}{n\rho(X_F)} \sup\left\{\rho(S) : S \in \mathfrak{S}_n(F)\right\}.$$

The real mathematical target:

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sup\left\{\rho(S) : S \in \mathfrak{S}_n(F)\right\}.$$

Extreme-aggregation Measure

Definition 4 (Extreme-aggregation measure)

An extreme-aggregation measure induced by a law-invariant risk measure ρ is defined as

$$\Gamma_\rho : \mathcal{X} \to [-\infty, \infty], \quad \Gamma_\rho(X_F) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup \left\{ \rho(S) : S \in \mathfrak{S}_n(F) \right\}.$$

Extreme-aggregation Measure

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- Γ_ρ quantifies the limit of ρ for worst-case aggregation under dependence uncertainty.
- Γ_ρ is a law-invariant risk measure.
- $\Gamma_\rho \geq \rho$.
- If ρ is subadditive then $\Gamma_\rho = \rho$.

Extreme-aggregation Measure

If ρ is (i) comonotonic additive, or (ii) convex and $\rho(0) = 0$, then

$$\Gamma_\rho(X_F) = \lim_{n\to\infty} \frac{1}{n} \sup\left\{\rho(S) : S \in \mathfrak{S}_n(F)\right\}.$$

- In the original definition of Γ_ρ it is actually "limsup" instead of "sup".
- Γ_ρ inherits monotonicity, cash-invariance, positive homogeneity, subadditivity, convexity, or zero-normalization from ρ if ρ has the corresponding properties.

Extreme-aggregation Measure

Question: given a non-subadditive risk measure ρ,

$$\text{Find } \Gamma_\rho$$

Extreme-aggregation Measure

Question: given a non-subadditive risk measure ρ,

Find Γ_ρ

- Motivating result (Wang and W., 2014):

$$\frac{\sup\{\mathrm{VaR}_p(S) : S \in \mathfrak{S}_n(F)\}}{\sup\{\mathrm{ES}_p(S) : S \in \mathfrak{S}_n(F)\}} \to 1.$$

Note that

$$\sup\{\mathrm{ES}_p(S) : S \in \mathfrak{S}_n(F)\} = n\mathrm{ES}_p(X_F),$$

leading to $\Gamma_{\mathrm{VaR}_p} = \Gamma_{\mathrm{ES}_p} = \mathrm{ES}_p$.

Distortion Risk Measures

Let h^* be the largest convex distortion function dominated by h.

Theorem: W., Bignozzi and Tsanakas, 2014, Preprint

Suppose ρ is a DRM with distortion function h, then $\Gamma_\rho = \rho^*$, where ρ^* is a coherent DRM with a distortion function h^*.

Distortion Risk Measures

Let h^* be the largest convex distortion function dominated by h.

Theorem: W., Bignozzi and Tsanakas, 2014, Preprint

Suppose ρ is a DRM with distortion function h, then $\Gamma_\rho = \rho^*$, where ρ^* is a coherent DRM with a distortion function h^*.

- ρ^* is the smallest coherent risk measure dominating ρ.
- Example: $\mathrm{VaR}_p^* = \mathrm{ES}_p$.
- For DRM, if $\rho(X_F) > 0$, then

$$\Delta^F(\rho) = \frac{\rho^*(X_F)}{\rho(X_F)}.$$

Convex Risk Measures

Theorem: W., Bignozzi and Tsanakas, 2014

Suppose ρ is a law-invariant convex risk measure, then

- Γ_ρ is a coherent risk measure.
- If ρ has the Fatou's property, then Γ_ρ is a coherent risk measure with representation

$$\Gamma_\rho = \sup_{h \in \mathcal{Q}} \left\{ \int \mathrm{ES}_p \mathrm{d}h(p) \right\},$$

where $\mathcal{Q} = \{h \in \mathcal{P}_I : a(h) > -\infty\}$, and a is the penalty function of ρ.

- Γ_ρ is the smallest coherent risk measure dominating ρ.

Shortfall Risk Measures

Theorem: W., Bignozzi and Tsanakas, 2014

Suppose ρ is a shortfall risk measure with loss function ℓ, then Γ_ρ is a coherent p-expectile, where

$$p = \lim_{x \to \infty} \frac{\ell'(x)}{\ell'(x) + \ell'(-x)}.$$

Regulatory Arbitrage

Regulatory arbitrage

- Write $X = \sum_{i=1}^{n} X_i$ and measure each X_i with ρ
- Compare $\rho(X)$ and $\sum_{i=1}^{n} \rho(X_i)$
- Make $\sum_{i=1}^{n} \rho(X_i)$ small: manipulation of risk
- Regulatory arbitrage: $\rho(X) - \sum_{i=1}^{n} \rho(X_i)$

Example of VaR

An example of VaR_p: for any risk $X > 0$, we can build

$$X_i = X\mathrm{I}_{A_i}, \;\; i = 1, \cdots, n$$

where $\{A_i\}$ is a partition of Ω and $\mathbb{P}(A_i) < 1 - p$. Then $\rho(X_i) = 0$. Therefore:

$$\sum_{i=1}^{n} X_i = X$$

and

$$\sum_{i=1}^{n} \rho(X_i) = 0.$$

Mathematical Treatment

Define

$$\Psi_\rho(X) = \inf\left\{\sum_{i=1}^n \rho(X_i) : n \in \mathbb{N},\ X_i \in \mathcal{X},\ i = 1, \dots, n,\ \sum_{i=1}^n X_i = X\right\}.$$

- $\Psi_\rho(X)$ is the least amount of capital requirement according to ρ if the risk X can be divided arbitrarily.
- $\Psi_\rho \le \rho$.
- $\Psi_\rho = \rho$ for subadditive risk measures.
- Regulatory arbitrage of ρ: $\rho(X) - \Psi_\rho(X)$.

Regulatory Arbitrage for VaR

Theorem: W., 2014, Working paper

For $p \in (0,1)$, $\Psi_{\mathrm{VaR}_p} = -\infty$.

- VaR is vulnerable to manipulation of risks.
- If ρ is a distortion risk measure, then Ψ_ρ is a coherent risk measure, but not necessarily a distortion.
- The regulatory arbitrage of VaR_p is infinity.

Regulatory Arbitrage for Convex Risk Measures

Theorem: W., 2014

If ρ is a law-invariant convex risk measure on L^∞ with penalty function v, then Ψ_ρ is a coherent risk measure with representation

$$\Psi_\rho = \sup_{h \in \mathcal{Q}} \left\{ \int \mathrm{ES}_p \mathrm{d}h(p) \right\},$$

where $\mathcal{Q} = \{h \in \mathcal{P}[0,1] : v(h) = 0\}$.

- Ψ_ρ is the largest coherent risk measure dominated by ρ.

Discussion

Coherence is indeed a natural property desired by a good risk measure. Even when a non-coherent risk measure is applied to a portfolio, its extreme behavior under dependence uncertainty leads to coherence.

Discussion

Coherence is indeed a natural property desired by a good risk measure. Even when a non-coherent risk measure is applied to a portfolio, its extreme behavior under dependence uncertainty leads to coherence.

When we allow arbitrary division of a risk, the extreme behavior also leads to coherence.

This contributes to the Basel question on ES versus VaR and partially supports the use of coherent risk measures.

Challenges

Some challenges and research directions:

- Discover new robustness properties for risk measures in practice; find risk measures that are more robust.
- New ways of backtesting ES and other coherent risk measures
- Quantifying model uncertainty for risk measures
- New statistical inference and computational methods for risk measures
- Extreme (catastrophic) events in risk management
- Risk measures in the presence of multiple securities

Challenges

Some mathematical research topics:

- Multi-period and continuous-time risk measures
- Set-valued, functional-valued, multi-dimensional risk measures
- Risk measures defined on stochastic processes
- Risk measures defined on data

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Thank you

Thank you for your kind attendance

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