Risk Aggregation with Dependence Uncertainty

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Seminar at ETH Zurich
November 5, 2013

Based on a series of joint work

Contents

Part I - Introduction

Risk and uncertainty:

- **Risk**: familiar; able to quantify; under control; quick response.
- **Uncertainty**: unfamiliar; difficult or impossible to quantify; beyond control; delayed response.

Part I - Introduction

Risk and uncertainty:

- **Risk**: familiar; able to quantify; under control; quick response.
- **Uncertainty**: unfamiliar; difficult or impossible to quantify; beyond control; delayed response.

Model risk: the risk of inappropriate modelling and misused quantitative tools.

- You think it is a risk but it is actually an uncertainty!

Risk aggregation

- $X_1, \cdots, X_n$ are random variables representing individual risks (one-period losses or profits).
- Aggregate position $S(\mathbf{X})$ associated with a risk vector $\mathbf{X} = (X_1, \cdots, X_n)$.
- The most commonly used aggregation function is $S = X_1 + \cdots + X_n$.

Challenges in dependence

- There is never perfect information. Statistical modelling and inference are needed.

	data	accuracy	modelling	calculation
marginal	rich	good	mature	easy
dependence	limited	poor	limited	heavy

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- Marginal $\rightarrow$ risk; dependence $\rightarrow$ uncertainty.

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- Marginal $\rightarrow$ risk; dependence $\rightarrow$ uncertainty.
- The logic of using parameters, such as covariance matrices, Spearman's rho and tail dependence coefficients, to model dependence in risk management is questionable.

Examples of model risk of dependence

Possibly misused modeling tools:

- Gaussian model.
- Conditional independence.
- Micro correlation.
- Independent increments.
- Behavior modeling.

April 23, 2013, S&P 500 index

What happend during those 10 minutes (1:07pm-1:16pm)?

Source: Yahoo finance

Part II - Dependence Uncertainty

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- $S = X_1 + \cdots + X_n$.
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- The joint structure of $X_1, \cdots, X_n$: unknown.
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Target: probabilistic behavior of S and/or risk measures of S.

Admissible risk class

Admissible risk class with uncertainty

For given univariate distributions $F_1, \cdots, F_n$, the admissible risk class (of marginals $F_1, \cdots, F_n$) is defined as

$$\mathfrak{S}_n(F_1, \cdots, F_n) = \{X_1 + \cdots + X_n : \ X_i \sim F_i, \ i = 1, \cdots, n\}.$$

Each $S \in \mathfrak{S}_n(F_1, \cdots, F_n)$ is called an admissible risk (of marginals $F_1, \cdots, F_n$).

A few remarks

- $\mathfrak{S}_n(\mathbf{F})$ is the set of all possible aggregate risks when the marginal distributions are accurately obtained but the joint distribution is unknown.

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- $\mathfrak{S}_n(\mathbf{F})$ is the set of all possible aggregate risks when the marginal distributions are accurately obtained but the joint distribution is unknown.
- The distribution of $S \in \mathfrak{S}_n(\mathbf{F})$ is determined by the copula of $X_1, \cdots, X_n$.
- This admissible risk class has some nice theoretical properties, such as convexity w.r.t. distribution, permutation/affine/law-invariance, completeness, robustness.

A few remarks

- In practice, people may have partial information about the joint structure, such as
  - individual risks are positively quadratic dependent;
  - individual risks are conditional independent;
  - some information on the copula of $\mathbf{X}$;
  - the covariance matrix is estimated accurately.

  In those cases, the possible aggregate risks are in a subset of $\mathfrak{S}_n(\mathbf{F})$.

Remark on Fréchet classes

A Fréchet class:

$$\mathfrak{F}_n(\mathbf{F}) := \{(X_1, \cdots, X_n) : \ X_i \sim F_i, \ i = 1, \cdots, n\}.$$

The difference between $\mathfrak{S}_n(\mathbf{F})$ and $\mathfrak{F}_n(\mathbf{F})$:

- The structure of $\mathfrak{F}_n(\mathbf{F})$ is marginal-independent, but $\mathfrak{S}_n(\mathbf{F})$ is marginal-dependent.
- The information contained in $\mathfrak{F}_n(\mathbf{F})$ is redundant.

Questions on admissible risk classes

- Probabilistically, what exactly are in the set $\mathfrak{S}_n(\mathbf{F})$?
  - For S with a given distribution F, is S in $\mathfrak{S}_n(\mathbf{F})$? Is there a viable characterization?
  - What is the boundary (in some sense) of $\mathfrak{S}_n(\mathbf{F})$?

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- Statistically, how can we conduct inference from data?
  - Traditional method: copula estimation - inaccurate, costly, provides information that are of no interest.
  - Direct estimation techniques: waste of marginal information.

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- Statistically, how can we conduct inference from data?
  - Traditional method: copula estimation - inaccurate, costly, provides information that are of no interest.
  - Direct estimation techniques: waste of marginal information.
- How can we use $\mathfrak{S}_n(\mathbf{F})$ to manage risks?
  - Assign a measure on $\mathfrak{S}_n(\mathbf{F})$? Risk $\Leftrightarrow$ uncertainty.
  - Extreme scenarios analysis?
  - Limited data regulation principles?

Part III - Extreme Scenarios

Extreme scenario questions for dependence uncertainty:

- Is a constant admissible?
- Convex ordering on admissible risks?
- Bounds for the distribution function of an admissible risk?

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- Convex ordering on admissible risks?
- Bounds for the distribution function of an admissible risk?

These three questions turn out to be closely connected, via the concept of completely mixable distributions.

A few remarks

- Extreme scenarios → coherent measure of model uncertainty defined in Cont (2006):

$$\mu_{\mathcal{Q}}(\rho) = \sup_{Q\in\mathcal{Q}} \rho^{Q}(S) - \inf_{Q\in\mathcal{Q}} \rho^{Q}(S).$$

- Research from the point of theoretical probability via a connection to mass-transportation can be found since early 80s, e.g. Rüschendorf (1982).
- A comprehensive overview on those topics can be found in the recent book Rüschendorf (2013).

Is a constant admissible?

- Basic observation: $\mathbb{E}[S]$ is a constant if $F_1, \cdots, F_n$ are L_1.
- Question: is a constant K, typically chosen as $\mathbb{E}[S]$, in $\mathfrak{S}_n(\mathbf{F})$?

Joint Mixability

Joint mixable distributions (W., Peng and Yang, 2013)

We say the univariate distributions $F_1, \cdots, F_n$ are jointly mixable (JM) if there exists $X_i \sim F_i$, $i = 1, \cdots, n$ such that $X_1 + \cdots + X_n$ is a constant. Equivalently,

$$\mathfrak{S}_n(F_1, \cdots, F_n) \cap \mathbb{R} \neq \varnothing.$$

Completely mixability

Completely mixable distributions (Wang and W., 2011)

We say the univariate distribution F is n-completely mixabe (CM) if there exists $X_1, \cdots, X_n \sim F$ such that $X_1 + \cdots + X_n$ is a constant. Equivalently,

$$\mathfrak{S}_n(F, \cdots, F) \cap \mathbb{R} \neq \varnothing.$$

Interpretation of CM and JM:

- CM or JM scenarios represent a perfectly hedged portfolio.
- It is an ideal case of negative correlation. It is a natural generalization of the counter-comonotonicity ($n = 2$).

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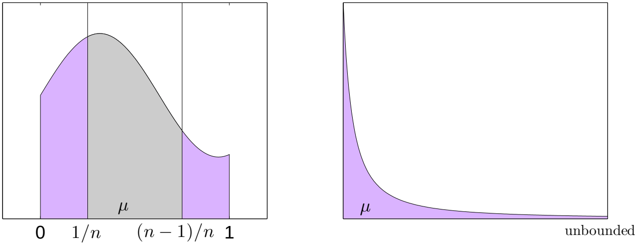
An open research area:

what distributions are CM/JM?

Most relevant results for CM:

- If F supported on $[a,b]$ with mean μ is n-CM, then the mean condition is **necessary**:

$$a+(b-a)/n \le \mu \le b-(b-a)/n.$$



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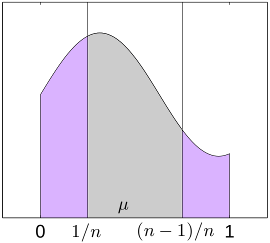

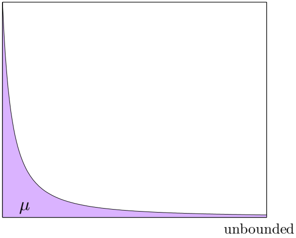

- The mean condition is **sufficient** for monotone densities.
- U[0,1] is n-CM for $n \geq 2$.

Some fully characterized families:

- Analytical proofs:
  - Rüschendorf and Uckelmann (2002): unimodal-symmetric densities.
  - Knott and Smith (2006) and Puccetti, Wang and W. (2012): radially symmetric distributions.
- Combinatorial proofs:
  - Wang and W. (2011): monotone densities.
  - Puccetti, Wang and W. (2012, 2013): concave densities; strictly positive densities.

Existing results for JM:

- Generalized mean condition.
- Second order condition: If $F_1, \cdots, F_n$ are JM with finite variance $\sigma_1^2, \cdots, \sigma_n^2$, then

$$\max_{i\in\{1,\cdots,n\}} \sigma_i \leq \frac{1}{2}\sum_{i=1}^{n} \sigma_i.$$

- W., Peng and Yang (2013): the variance condition is sufficient for normal.
- Wang and W. (2013a, preprint): the variance condition is sufficient for uniform; elliptical; and unimodal-symmetric densities.

Mysteries of CM (JM)

- Uniqueness of the center?
- Unimodal densities and other types?
- Characterization?
- Asymptotic behavior ($n \to \infty$)?

Convex ordering bounds

- We assume the individual risks are on $\mathbb{R}_+$ and are L_1 (finite mean).
- Since $\mathbb{E}[S]$ is fixed, the most interesting property is the **convex order** of $\mathfrak{S}_n(\mathbf{F})$:

For $X, Y \in L_1$, if $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for all convex functions $g : \mathbb{R} \to \mathbb{R}$, then we say $X \prec_{\mathrm{cx}} Y$.

- In economics, the term second order stochastic dominance is more often used.

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  - the variance of aggregation, European basket option prices, realized variance options;
  - stop-loss premiums, losses with limits/deductibles.

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- Directly connects to bounds on the Value-at-Risk and optimal mass transportation problems.
- Mathematically nice and tractable.

Well-known results on this question (late 70s):

- The convex order upper bound is obtained by the **comonotonic scenario**: for $S \in \mathfrak{S}_n(\mathbf{F})$,

$$S \prec_{\mathrm{cx}} F_1^{-1}(U) + \cdots + F_n^{-1}(U)$$

where U has a uniform distribution on $[0, 1]$.

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- The infimum:
  - Known for $n = 2$: **counter-monotonic scenario** for $S \in \mathfrak{S}_2(F_1, F_2)$:

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  - Mysterious for $n \geq 3$ in general.
- All the above results are **marginal-independent**.

Connection between CM/JM distribution and convex ordering lower bound for $n \geq 3$:

- If $F_1, \cdots, F_n$ are JM, then $\mathbb{E}[S]$ is in $\mathfrak{S}_n(F_1, \cdots, F_n)$, and thus it is the convex minimal element.
- CM/JM scenario is a natural generalization of the counter-comonotonicity.
- Please note that the optimal structure is **marginal-dependent**. (I believe it is the reason why major progresses on this problem were delayed till recently.)

Existence

A surprising fact: for $n \geq 3$, the set $\mathfrak{S}_n(\mathbf{F})$ may not contain a convex ordering minimal element.

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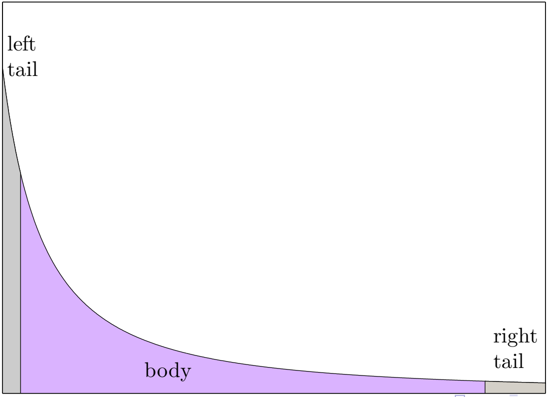
- Identical and monotone marginal densities: analytical results obtained in Wang and W. (2011).
- General marginal densities on $\mathbb{R}_+$: Bernard, Jiang and W. (2013, preprint).
- To obtain a convex minimal element, we try to enhance a density concentration (make S as close to a constant as possible).

A few remarks for main results in Bernard, Jiang and W. (2013, preprint):

- Optimal structure for homogeneous marginals: tails - mutual exclusivity; body - complete mixability.



- Analytical formulas for the lower bound on $\mathrm{TVaR}_p(S)$ and $\mathbb{E}[g(S)]$ are available.
- Lower bounds for heterogeneous marginals are obtained:
  - not sharp in general, but quite accurate according to numerical results;
  - the fact $\mathfrak{S}_n(F_1, \cdots, F_n) \subset \mathfrak{S}_n(F, \cdots, F)$ is used, where $F = \frac{1}{n}\sum_{i=1}^n F_i$.

Bounds on the distribution function

- Given marginal distributions, what is the maximum possible distribution function of S (a special case of a question raised by A. N. Kolmogorov)?

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- The question: given $F_1, \cdots, F_n$ and $s \in \mathbb{R}$, find

$$\sup_{S \in \mathfrak{S}_n(F_1,\cdots,F_n)} \mathbb{P}(S \leq s) \text{ and } \inf_{S \in \mathfrak{S}_n(F_1,\cdots,F_n)} \mathbb{P}(S \leq s).$$

Equivalent question in risk management:

- Given $F_1, \cdots, F_n$ and $\alpha \in (0,1)$, find

$$\sup_{S \in \mathfrak{S}_n(F_1,\cdots,F_n)} \mathrm{VaR}_\alpha(S) \text{ and } \inf_{S \in \mathfrak{S}_n(F_1,\cdots,F_n)} \mathrm{VaR}_\alpha(S).$$

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- It is the best/worst scenario risk measure with confidence in marginal information.
- The usage of VaR in risk management is debatable for incoherence (non-subadditivity in particular) but still quite widely used.
- Very hard to solve analytically.
- What is done in the practice of operational risk: model marginal, add them up, and discount to 70%-90% due to *unjustified* diversification benefit.

Some literature

- Makarov (1981): $n = 2$.
- Rüschendorf (1982): independently solved $n = 2$.
- Identical marginals:
  - Rüschendorf (1982): dual representation; uniform and binomial cases.
  - Denuit, Genest and Marceau (1999): non-sharp standard bound.
  - Embrechts and Puccetti (2006): dual bounds.
  - W., Peng and Yang (2013): sharp bounds for homogeneous tail monotone densities based on CM.
  - Puccetti and Rüschendorf (2013): sharpness of dual bounds, equivalent to a CM condition.
- Embrechts, Puccetti and Rüschendorf (2013): numerical algorithm and general discussion.

Between VaR and convex ordering bounds

Suppose $F_1, \cdots, F_n$ are continuous distributions.

- $F_{i,a}$ for $a \in (0,1)$ is the conditional distribution of F_i on $[F_i^{-1}(a), \infty)$;
- F_i^a for $a \in (0,1)$ is the conditional distribution of F_i on $(-\infty, F_i^{-1}(a))$.

Convex ordering lower bound and bounds on VaR

Theorem 1 (Bernard, Jiang and W. (2013))

(a) *Suppose* S_a *is a convex ordering minimum element in* $\mathfrak{S}_n(F_{1,a},\cdots,F_{n,a})$ *for* $a\in(0,1)$*, then* $\sup_{S\in\mathfrak{S}_n(F_1,\cdots,F_n)}\mathrm{VaR}_a(S)=\operatorname{ess\,inf} S_a$.

(b) *Suppose* S_a *is a convex ordering minimum element in* $\mathfrak{S}_n(F_1^a,\cdots,F_n^a)$ *for* $a\in(0,1)$*, then* $\inf_{S\in\mathfrak{S}_n(F_1,\cdots,F_n)}\mathrm{VaR}_a(S)=\operatorname{ess\,sup} S_a$.

A few remarks:

- Finding convex ordering minimal element implies worst and best elements for VaR.
- The worst VaR only depends on the tail behavior, hence extra information on convariance/correlation may or may not affect its value.
- Bernard, Rüschendorf and Vanduffel (2013, preprint): VaR bounds with variance constraint on S.

Part IV - Asymptotic Behavior

- Look at $S_n \in \mathfrak{S}_n(\mathbf{F})$, $\mathbf{F} = (F, \cdots, F)$, F having mean μ.
- When F has finite second moment, we have looked at

$$\overline{V}_n = \mathrm{Var}(S_n) \text{ and } \underline{V_n} = \inf_{S_n \in \mathfrak{S}_n(\mathbf{F})} \mathrm{Var}(S_n).$$

- What if $n \to \infty$?

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$$V_n = \mathrm{Var}(S_n) \text{ and } \underline{V_n} = \inf_{S_n \in \mathfrak{S}_n(\mathbf{F})} \mathrm{Var}(S_n).$$

- What if $n \to \infty$?
  - iid case: $V_n = O(n)$.
  - comonotonic case: $V_n = O(n^2)$.
  - what about most negative correlated case $\underline{V_n}$?

Variance reduction

Theorem 2 (Wang and W. (2013b, preprint))

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- A stronger result: there exists a sequence $X_i,\ i \in \mathbb{N}$ from F such that $|S_n - n\mu| \leq Z$ a.s. for some Z which does not depend on n.
- For some F this $O(1)$ is sharp, i.e. $\underline{V_n} \not\to 0$.

Asymptotic CM

Theorem 3 (Puccetti, Wang and W. (2013))

Suppose F is supported in a finite interval with a strictly positive density function, then there exists $N \in \mathbb{N}$ such that F is n-CM for all $n \geq N$.

Asymptotically every distribution is (almost) CM.

Asymptotic equivalence

Theorem 4

Under some conditions on F, for all $a \in (0,1)$

$$\frac{\sup_{S \in \mathfrak{S}_n(F,\cdots,F)} \mathrm{VaR}_a(S)}{\sup_{S \in \mathfrak{S}_n(F,\cdots,F)} \mathrm{TVaR}_a(S)} \to 1.$$

Worst VaR and worst TVaR (ES) are asympototically equivalent.

- Puccetti and Rüschendorf (2013): F is continuous, satisfies a conditional CM condition.
- Puccetti, Wang and W. (2013): F is continuous and has strictly positive density based on CM.
- Wang and W. (2013b, preprint): F is arbitrary, and no CM involved.

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- The same asymptotic equivalence holds for inhomoegenous marginals with very weak conditions on the marginal distributions.

Table : Values (rounded) for best- and worst VaR and ES for a homogeneous portfolio with d Pareto(2) risks; $\alpha = 0.999$.

$\theta = 2$	$d = 8$	$d = 56$
Best VaR	31	53
Best TVaR	145	472
Comonotonic VaR	245	1715
Worst VaR	465	3454
Worst TVaR	498	3486

In practice some people would use about $\mathrm{VaR}_\alpha(S) \approx 200$ for $d = 8$ as the *conservative* capital reserve.

Shape problem

Question. Let F, G be any two univariate distributions. Can you find random variables X_i, $i \in \mathbb{N}$ from F such that $(S_n - a_n)/b_n \xrightarrow{d} G$ for some real sequences a_n, b_n?

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I think the answer is positive. The message is:

The marginal constraint is weak compared to the dependence uncertainty. If you only assume known marginals, you can end up with **anything.**

Part V - Challenges

- Theoretical results are basically unavailable for heterogeneous marginal distributions.
- Many unsolved mathematical problems.
- Applications in quantitative risk management.

Mathematical challenges

- Develop more classes of CM/JM distributions.
- Find sharp convex bounds for non-identical marginal distributions.
- Sufficient conditions for the existence of convex ordering minimal element in an admissible risk class?
- Improve numerical algorithms such as the Rearrangement Algorithm in Embrechts, Puccetti and Rüschendorf (2013).

Final remarks

- Practical risk management?
- Dynamic process?
- I expect connection with statistics and data science.
  - Modelling aggregate risks via estimating dependence structure may not be the best idea to study risk aggregation.
- Rather immature ideas; discussions are very much welcome.

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Thank you for your attention

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