Worst-case Range Value-at-Risk with Partial Information

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Abstract

In this paper, we study the worst-case scenarios of a general class of risk measures, the Range Value-at-Risk (RVaR), in single and aggregate risk models with given mean and variance, as well as symmetry and/or unimodality of each risk. For different types of partial information settings, sharp bounds for RVaR are obtained for single and aggregate risk models, together with the corresponding worst-case scenarios of marginal risks and the corresponding copula functions (dependence structure) among them. Different from the existing literature, the sharp bounds under different partial information settings in this paper are obtained via a unified method combining convex order and the recently developed notion of joint mixability. As particular cases, bounds for Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) are derived directly. Numerical examples are also provided to illustrate our results.

Keywords: model uncertainty, risk aggregation, Range Value-at-Risk, Value-at-Risk, Tail Value-at-Risk, convex order.

1 Introduction

1.1 Problem formulation and related literature

Quantification of risky positions held by a financial institution under model uncertainty is of crucial importance from both viewpoints of external regulation and internal management. Very

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often, the problem of interest is of the following type: to find

$$\begin{aligned} &\sup \rho(X_1 + X_2 + \cdots + X_n) \\ &\text{over} \quad X_i \in \mathcal{P}_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

where ρ is a risk measure, and for each $i = 1, \ldots, n$ the set $\mathcal{P}_i$ is a class of random variables with some given partial distributional information. With a partially specified model, the value of the risk measure ρ varies in a range over the set of all possible models. The largest value in such a range is referred to a *worst-case value*, and the corresponding model is called a *worst scenario*. In this paper, the distributional information we consider includes specified moments such as mean and variance, and descriptive information such as symmetry and unimodality.

For single risk models, that is, $n = 1$ in (1), an early source is Royden (1953). There are more developments in the recent few decades. Kaas and Goovaerts (1986) calculated the distribution-free bound of $\mathbb{P}(X \leq t)$ under partial information including the mean and the variance with the *duality method*, and later De Schepper and Heijnen (2010) extended the bound of $\mathbb{P}(X \leq t)$ with more partial information on the mode of the distribution besides the original mean and variance. See Hurlimann (2002) for applications of the problem in actuarial science. Popescu (2005) incorporated symmetry into the problem and used an operational method called *semidefinite programming* to determine the best-possible bounds on $\mathbb{P}(X \leq t)$ with additional assumption including symmetry and unimodality, and He et al. (2010) obtained upper bounds on $\mathbb{P}(X \leq t)$ with partial information of the first four moments of X by using the same method. A generalized semidefinite programming method to calculate the probability of a random vector falling outside a polytope based on moment and shape information is recently given in Van Parys et al. (2016). For a single random variable X with partial information, literature of finding distribution-free bounds on quantities of the form $\mathbb{E}[\phi(X)]$ for some functions ϕ dates back to De Vylder (1982) and De Vylder and Goovaerts (1982).

For aggregate risk models, that is, $n > 1$ in (1), finding bounds for quantities related to the sum of random variables with the knowledge of marginal distributions is typically called the *Fréchet problem*, where the complete uncertainty of the dependence is typically assumed. For recent research on Fréchet problems for VaR and convex risk measures, see Embrechts et al. (2013), Wang et al. (2013), Bernard et al. (2014) and Cai et al. (2017). For VaR bounds with partial dependence information in addition to the marginal information, see Bernard et al. (2016b, 2017a), Bernard and Vanduffel (2015) and Puccetti et al. (2016, 2017), amongst others. We refer to Embrechts et al. (2014) for general discussions on this problem. Besides the assumption of the complete uncertainty dependence, there is also recent literature studying the optimization problem (worst-case) of the risk measures under partial information with some information of dependence; see Zhu and Fukushima (2009), Zymler et al. (2013) and Cambou and Filipovic (2017).

Diffierent from the existing literature which mainly focus on worst-case values of the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR), this paper discusses a more general class of risk measures: the Range Value-at-Risk (RVaR), which was proposed by Cont et al. (2010) as a generalization of VaR and TVaR.

Robustness of risk measures is a central issue in recent years. Heyde et al. (2006) is one of the earliest papers that discusses the robustness of risk measures. Heyde et al. (2006), Cont et al. (2010) and Kou and Peng (2016) pointed out that VaR is more robust than TVaR, in the sense that VaR is continuous with respect to convergence in distribution at random variables with a continuous quantile function, whereas TVaR is not continuous at any random variables in the same sense. Embrechts et al. (2014) contains a comprehensive discussion on recent issues related to the robustness of VaR and TVaR; see Krätschmer et al. (2014) for various notions of robustness for general convex risk measures. RVaR, which can be seen as a bridge between VaR and RVaR, is robust in the sense that it is continuous with respect to weak convergence of random variables, whereas neither VaR or TVaR is continuous in general. Bignozzi and Tsanakas (2016) studied residual estimation risk for various risk measures including RVaR. Given the great generality and nice properties of RVaR, we discuss RVaR with the model uncertainty of the underlying risks in this paper, and the results can be naturally transferred to VaR and TVaR by taking limits.

1.2 Main contribution of this paper

For both single risk models and aggregate risk models, this paper derives worst-case values and worst-case scenarios for RVaR under partial information assumption.

For the partial information settings, we address model uncertainty by assuming that the mean, variance, and additionally symmetry and/or unimodality of each risk are known. Note that the mean and the variance are among the most traditional non-parametric statistics, thus the partial information assumption is fairly reasonable from the practical view. This paper only considers the first two moments for mathematical tractability, which is a standard setup in the distributional optimization literature (Van Parys et al., 2016). On the other hand, the shape assumptions such as unimodality and symmetry are also reasonable for that most parametric univariate distributions are symmetric or unimodal. For example, exponential, Beta, Gamma, log-normal, uniform Chisquare, Pareto and Weibull distribution are unimodal distributions, and normal, Cauchy, logistic and student's t-distribution are unimodal and symmetric distributions. Moreover, in the modeling of credit portfolio's Loss Given Default (LGD), LGD is often assumed to be a unimodal distribution on $[0, 1]$ (Gupton and Stein, 2005). Some unimodal and symmetrical distributions are completely or jointly mixable (Wang and Wang, 2011, 2016), and this fact leads to the analytical solutions for risk aggregation in this paper.

For single risk models, we derive worst-case values and worst-case scenarios for RVaR under four different settings of partial information. Different from most of the literature in which different methods are designed for various settings, we establish a unified approach for all settings of partial information and for all RVaR risk measures including VaR and TVaR by using stochastic orders. For some recent work using stochastic orders to derive bounds on risk measures with partial information, please see Jakobsons and Vanduffel (2015) and Bernard et al. (2016a, 2017b).

For aggregate risk models, we consider a setting as in (1) by combining marginal uncertainty and dependence uncertainty. Worst-case values for RVaR and their corresponding dependence structures are obtained. The results mainly rely on the recently developed notion of joint mixability. Different from the classical Fréchet problem where marginal distributions are assumed to be known, we only assume partial information on each individual risk. This, in particular, addresses one question proposed by Embrechts et al. (2014): the combination of marginal (statistical) uncertainty and dependence uncertainty. Our results for the aggregate risk model show that if there is no single risk whose standard deviation dominates the sum of the other risks in the portfolio, then the worst-case scenario for RVaR in risk aggregation can be obtained via the worst-case distributions of individual risks, combined with a dependence structure of conditional joint mixability, which is consistent with the VaR moment bounds in Wang et al. (2013) and Puccetti and Rüschendorf (2013); for the case that a single risk has a dominating standard deviation comparing to other risks, we show that the worst-case dependence follows a specific structure whose special bivariate case was considered in Embrechts et al. (2005). These conservative bounds obtained in this paper generalize the existing results in the literature and provide a valuable reference to help make financial and insurance decisions. Different from the traditional approaches of duality method and semidefinite programming, our method is based on the stochastic comparison, and we provide analytical formulas for the worst-case values of RVaR. The worst scenario and the corresponding distribution of the underlying risk can be determined simultaneously.

The rest of the paper is organized as follows. In Section 2, we give our main results on the worst scenarios of RVaR on single risks. In Section 3, we analyze the worst scenarios of RVaR on aggregate risks via basing on the results in Section 2. Detailed proofs of the main results are put in Section 4. A conclusion is drawn in Section 5.

2 Worst scenarios for RVaR of single risks

2.1 Definitions and some notations

For a random variable X, its right-continuous VaR is defined as

$$\mathrm{VaR}_\alpha[X] = \inf\{x \in \mathbb{R} \,|\, \mathbb{P}(X \le x) > \alpha\}, \quad \alpha \in (0,1).$$

The difference between the right- and left-continuous versions of VaR is inessential and usually they are indistinguishable in practice; see Embrechts and Hofert (2013) for details on generalized quantiles. Due to technical convenience we use the right-continuous version in this paper.

Range-VaR (RVaR) was proposed in Cont et al. (2010) as a robust risk measure, defined as

$$\mathrm{RVaR}_{\alpha,\beta}[X] = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathrm{VaR}_u[X]du, \quad 0 < \alpha < \beta < 1.$$

Range-VaR includes TVaR and VaR as its limiting cases. Denote by L^p, $p \in [0,\infty]$ the set of random variables with finite p-th moment. The TVaR of a risk $X \in L^1$ is defined as

$$\mathrm{TVaR}_\alpha[X] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u[X]du, \quad \alpha \in (0,1).$$

Obviously

$$\mathrm{RVaR}_{\alpha,1}[X] := \lim_{\beta\uparrow 1} \mathrm{RVaR}_{\alpha,\beta}[X] = \mathrm{TVaR}_\alpha[X].$$

If $\mathbb{P}(X = \mathrm{VaR}_\alpha[X]) = 0$, one may write $\mathrm{TVaR}_\alpha[X] = \mathbb{E}[X|\, X > \mathrm{VaR}_\alpha[X]]$. As for VaR, by the right-continuity of $\mathrm{VaR}_\alpha[X]$, one has

$$\lim_{\beta\downarrow\alpha} \mathrm{RVaR}_{\alpha,\beta}[X] = \mathrm{VaR}_\alpha[X].$$

Also note that the choice of right- or left-continuous version of VaR is irrelevant to the quantity of TVaR and RVaR, since the two versions of VaR for a risk only differ at countably many points.

We say a random variable X (or its corresponding distribution) is *unimodal* if there is a constant

m such that its distribution function F_X is convex on $(-\infty, m)$ and concave on (m, ∞), and we call m the *mode* of X. In particular, if X is unimodal and continuous, then its density function f_X is increasing on $(-\infty, m)$ and decreasing on (m, ∞). A unimodal random variable can be written as the mixture of a point mass at its mode and a continuous unimodal random variable with the same mode (see for instance Feller, 1971, Section 5.9). On the other hand, we say a random variable X (or its corresponding distribution) is *symmetric* if there is a constant m such that $\mathbb{P}(X \le x) = \mathbb{P}(X \ge 2m - x)$ for any $x \in \mathbb{R}$, and we call m the *symmetric center*. Obviously, if X is unimodal-symmetric (unimodal and symmetric), then its mode coincides with its symmetric center.

For $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$, we denote by $V(\mu, \sigma)$ the set of random variables with mean μ and variance σ^2, and denote by $V_S(\mu, \sigma)$, $V_U(\mu, \sigma)$, $V_{US}(\mu, \sigma)$ the sets of symmetric, unimodal, and unimodal-symmetric random variables in $V(\mu, \sigma)$, respectively. We omit μ and σ when $\mu = 0$ and $\sigma = 1$; for instance V_U represents $V_U(0, 1)$.

2.2 Main results for the worst scenarios on single risk

Our aim is to determine

$$\sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X]\,, \tag{2}$$

where $\mathcal{V}(\mu, \sigma)$ is one of $V(\mu, \sigma)$, $V_S(\mu, \sigma)$, $V_U(\mu, \sigma)$ or $V_{US}(\mu, \sigma)$. Denote

$$\overline{\mathrm{RVaR}}_{\alpha,\beta}^{\mathcal{V}(\mu,\sigma)} = \sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X]\,.$$

It is easy to verify that all the upper bounds are linear functions of μ and σ, i.e.

$$\overline{\mathrm{RVaR}}_{\alpha,\beta}^{\mathcal{V}(\mu,\sigma)} = \mu + \sigma \cdot \overline{\mathrm{RVaR}}_{\alpha,\beta}^{\mathcal{V}(0,1)}$$

for $\mathcal{V}(\mu, \sigma) \in \{V(\mu, \sigma), V_S(\mu, \sigma), V_U(\mu, \sigma), V_{US}(\mu, \sigma)\}$. Note that the problem (2) only concerns the distribution of random variable in $\mathcal{V}(0, 1)$. If a random variable $X^* \in \mathcal{V}(\mu, \sigma)$ satisfies $\mathrm{RVaR}_{\alpha,\beta}[X^*] = \overline{\mathrm{RVaR}}_{\alpha,\beta}^{\mathcal{V}(\mu,\sigma)}$, we say that its distribution F_{X^*} is a *worst-case distribution* relative to $\mathcal{V}(\mu, \sigma)$.

First we introduce some special families of random variables for the candidates of the worst-case distributions. Denote $x \in \mathbb{R} \mapsto U(x; a, b)$ as the distribution function of a uniform random variable over $[a, b]$. Note that for some $a \in \mathbb{R}$, the indicator function $I[x \ge a], x \in \mathbb{R}$ is the distribution function of a point mass concentrated at a. Using the above two functions, we define the following mixture distributions (see Figure 1 below)

$$
\begin{aligned}
G_\theta(x) &= \theta I\left[x \geq -\sqrt{\tfrac{1-\theta}{\theta}}\right] + (1-\theta) I\left[x \geq \sqrt{\tfrac{\theta}{1-\theta}}\right], \quad \theta \in (0,1);\\
G^S_\theta(x) &= (1-\theta) I\left[x \geq -\sqrt{\tfrac{1}{2(1-\theta)}}\right] + (2\theta-1) I[x \geq 0] + (1-\theta) I\left[x \geq \sqrt{\tfrac{1}{2(1-\theta)}}\right], \quad \theta \in (\tfrac{1}{2},1);\\
G^U_\theta(x) &= \tfrac{3-3\theta}{2} U\left(x; -\sqrt{\tfrac{9-9\theta}{9\theta-1}}, \tfrac{1+3\theta}{\sqrt{(1-\theta)(9\theta-1)}}\right) + \tfrac{3\theta-1}{2} I\left[x \geq -\sqrt{\tfrac{9-9\theta}{9\theta-1}}\right], \quad \theta \in [\tfrac{2}{3},1);\\
G^{US}_\theta(x) &= (3-3\theta) U\left(x; -\sqrt{\tfrac{1}{1-\theta}}, \sqrt{\tfrac{1}{1-\theta}}\right) + (3\theta-2) I[x \geq 0], \quad \theta \in [\tfrac{5}{6},1),
\end{aligned}
$$

where $x \in \mathbb{R}$. Here G_θ is a two-point distribution, G^S_θ is a three-point distribution, and G^U_θ and G^{US}_θ are mixtures of uniform and one-point distributions.

The distributions G_θ, G^S_θ, G^U_θ and G^{US}_θ are associated with the sets V, V_S, V_U and V_{US}, respectively, as summarized in the following lemma. Figure 1 illustrates the above four families of distributions through their quantile functions.

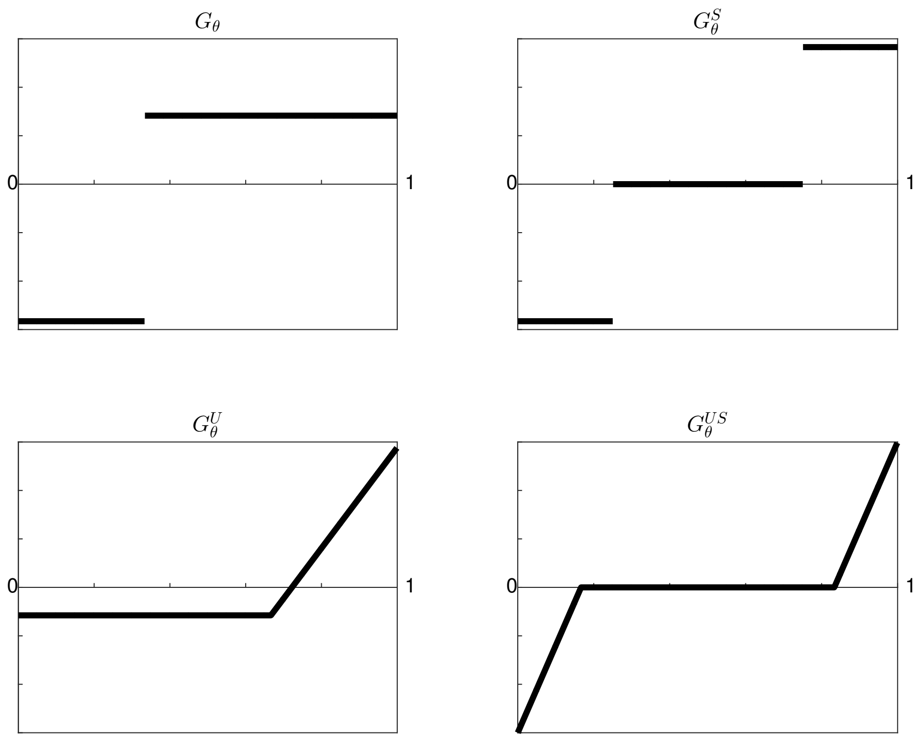

Figure 1: Quantile functions of G_θ, G^S_θ, G^U_θ and G^{US}_θ.

The following lemma can be checked directly and the proof is omitted.

Lemma 1. *We have that* $X \in V(0,1)$ *if* $X \sim G_\theta$ *for some* $\theta \in (0,1)$*,* $X \in V_S(0,1)$ *if* $X \sim G_\theta^S$ *for some* $\theta \in (\frac{1}{2},1)$*,* $X \in V_U(0,1)$ *if* $X \sim G_\theta^U$ *for some* $\theta \in [\frac{2}{3},1)$*, and* $X \in V_{US}(0,1)$ *if* $X \sim G_\theta^{US}$ *for some* $\theta \in [\frac{2}{3},1)$*.*

The following theorem gives the worst-case values of RVaR and their corresponding worst-case distributions. The proof of Theorem 1 will be given in Section 4.

Theorem 1. *(a) For* $(\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+$*, we have*

$$\overline{\mathrm{RVaR}}_{\alpha,\beta}^{\mathcal{V}(\mu,\sigma)} = \begin{cases} \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\,, & \mathcal{V} = V,\ 0 < \alpha < \beta < 1; \\ \mu + \sigma\sqrt{\frac{1}{2(1-\alpha)}}\,, & \mathcal{V} = V_S,\ \frac{1}{2} < \alpha < \beta < 1; \\ \mu + \sigma\sqrt{\frac{8}{9(2-\alpha-\beta)} - 1}\,, & \mathcal{V} = V_U,\ \frac{5}{6} \le \alpha < \beta < 1; \\ \mu + \sigma\sqrt{\frac{4}{9(2-\alpha-\beta)}}\,, & \mathcal{V} = V_{US},\ \frac{5}{6} \le \alpha < \beta < 1. \end{cases} \tag{3}$$

(b) The first equality holds if $X \sim G_\alpha\left(\frac{x-\mu}{\sigma}\right)$*, the second equality holds if* $X \sim G_\alpha^S\left(\frac{x-\mu}{\sigma}\right)$*, the third equality holds if* $X \sim G_{\alpha+\beta-1}^U\left(\frac{x-\mu}{\sigma}\right)$*, and the last equality holds if* $X \sim G_{\alpha+\beta-1}^{US}\left(\frac{x-\mu}{\sigma}\right)$*.*

From the above theorem we can see that for the two sets $\mathcal{V} = V(\mu,\sigma)$ and $\mathcal{V} = V_S(\mu,\sigma)$ the worst-case distributions are discrete distributions, and for the two sets $\mathcal{V} = V_U(\mu,\sigma)$ and $\mathcal{V} = V_{US}(\mu,\sigma)$ the worst-case distributions are the mixtures of uniform and one-point distributions. The restriction $\alpha \ge 5/6$ is relevant in practice, for instance VaR is often considered for $\alpha \ge 95\%$. By setting $\beta \downarrow \alpha$ in Theorem 1, sharp upper bounds for VaR are obtained, and letting $\beta \uparrow 1$, sharp upper bounds for TVaR are obtained.

Remark 1. In the cases of $X \in V(\mu,\sigma)$, the $\mathrm{VaR}_\alpha(X)$ bound is straightforward from the Cantalli inequality. For $X \in V_S(\mu,\sigma)$ and $X \in V_{US}(\mu,\sigma)$, the best-possible bounds on $\mathbb{P}(X \le t)$ with a certain $t \in \mathbb{R}$ have been provided in Popescu (2005), from which the corresponding VaR bounds under the same settings can be also derived. As far as we are aware, bounds on risk measures over $V_U(\mu,\sigma)$ have not been considered in existing literature. Note that the set $V_U(\mu,\sigma)$ includes Pareto, Log-normal, Gamma and many other common distributions in risk management, hence the study of bounds for $V_U(\mu,\sigma)$ provide great complement to those for $V(\mu,\sigma)$ and $V_S(\mu,\sigma)$.

Remark 2. The results in Theorem 1 are intuitive. With the information of the mean and variance, maximizing VaR requires a flat quantile function in the right tail and moreover, for a given mean, minimum variance is then obtained by making the quantile function flat on the other part of the support too. Combining these two features leads to two-point distributions when maximizing VaR with mean and variance given. Under the assumption of unimodality, the distribution/quantile function needs one concave part and one convex part. It is then intuitive to find that the worst-case distributions should behave as a mixture of a uniform distribution and a single-point distribution.

Remark 3. Bernard et al. (2016a) derive bounds on VaR, a limit case of RVaR, when higher order moment information is available; see also Bernard et al. (2017b) for a related application. With higher moments available, bounds on VaR are derived using moment inequalities (rather than equalities). Without assuming unimodality, our results can be naturally extended to the case of higher order moment inequalities, since the worst-case distributions are a combination of point-masses. The case with unimodality and higher moment information may be more complicated, and we leave it for future work.

2.3 Some remarks on VaR and TVaR

Since VaR_α and TVaR_α are limits of $\mathrm{RVaR}_{\alpha,\beta}$ as $\beta \downarrow \alpha$ and $\beta \uparrow 1$, respectively, bounds on VaR and TVaR can be directly derived from Theorem 1 by taking limits, as long as we justify the exchange of the order of limit and supremum, as in the following lemma. The proof will be given in Section 4.

Lemma 2. *For $(\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+$, $\mathcal{V} \in \{V, V_S, V_U, V_{US}\}$ and $\alpha \in (5/6, 1)$, we have*

$$\sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{VaR}_\alpha[X] = \lim_{\beta \downarrow \alpha} \sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X],$$

$$\sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{TVaR}_\alpha[X] = \lim_{\beta \uparrow 1} \sup_{X \in \mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X].$$

A summary of the VaR and TVaR bounds are given in Table 1. Notice the fact that if the underlying risk is not assumed to be unimodal, i.e. $X \in V(\mu,\sigma)$ or $X \in V_S(\mu,\sigma)$, the VaR bound is equal to the corresponding TVaR bound; while in the other two cases that the underlying risk is assumed to be unimodal, VaR bound is less than the corresponding TVaR bound.

A numerical report of the sharp bounds for VaR and TVaR are plotted in Figure 2. Intuitively, more restrictions on probabilistic information would lead to a smaller bound. Comparing the worst-case values for VaR and TVaR under different settings in Figure 2, it shows that the additional restriction of unimodality has a significant impact on the sharp VaR bounds, but the impact on the sharp TVaR bounds is minor. However, the additional restriction of symmetry has a relatively

Table 1: The worst-case values (WV) of VaR and TVaR with partial information.

Restriction	Property	WV of $\mathrm{VaR}_\alpha[X]$	WV of $\mathrm{TVaR}_\alpha[X]$
$X \in V(\mu,\sigma)$	general	$\mu+\sigma\cdot\sqrt{\frac{\alpha}{1-\alpha}}$	$\mu+\sigma\cdot\sqrt{\frac{\alpha}{1-\alpha}}$
$X \in V_S(\mu,\sigma)$	symmetric	$\mu+\sigma\cdot\sqrt{\frac{1}{2(1-\alpha)}}$	$\mu+\sigma\cdot\sqrt{\frac{1}{2(1-\alpha)}}$
$X \in V_U(\mu,\sigma)$	unimodal	$\mu+\sigma\cdot\sqrt{\frac{4}{9(1-\alpha)}-1}$	$\mu+\sigma\cdot\sqrt{\frac{8}{9(1-\alpha)}-1}$
$X \in V_{US}(\mu,\sigma)$	unimodal-symmetric	$\mu+\sigma\cdot\sqrt{\frac{2}{9(1-\alpha)}}$	$\mu+\sigma\cdot\sqrt{\frac{4}{9(1-\alpha)}}$

significant impact on both VaR and TVaR worst-case values.

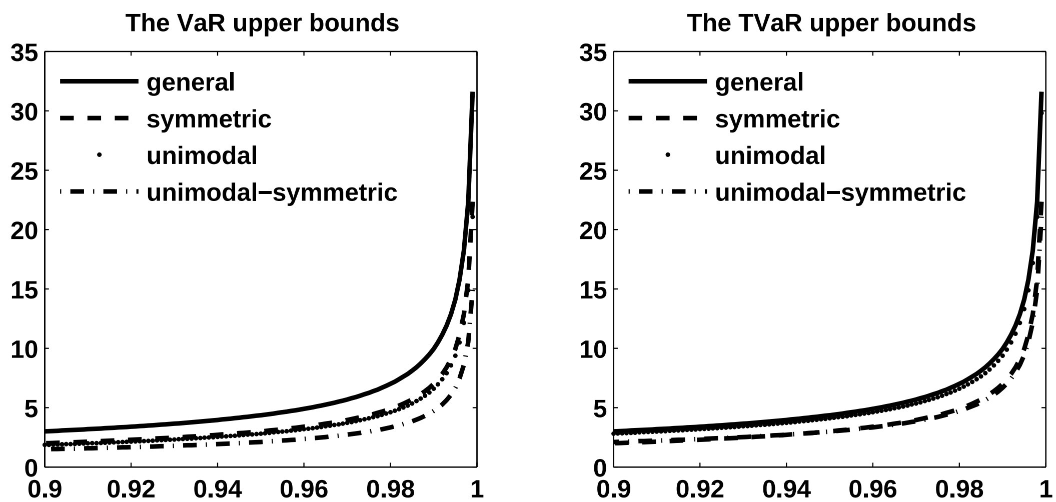

Figure 2: The VaR worst-case values (left) and TVaR worst-case values (right) under different settings of partial information, in which the underlying risks are assumed to be zero mean and unit variance.

As an application, we show standardized (setting mean 0 and variance 1) VaR of some parametric families and the corresponding worst-case values in Table 2. We also compute ratios of the upper bound over each parametric VaR. This ratio measures the magnitude of model uncertainty caused by assuming specific parametric models. From Table 2, we can see that this ratio is around 2 and 3, which gives an theoretical support to the regulations by the Basel Committee where 3 is set as the minimum *multiplication factor* in Basel Committee (1996).

3 Worst scenarios for RVaR of aggregate risks

This section discusses worst scenarios for RVaR on aggregate risk models. To illustrate the dependence structure among individual risks explicitly, we use the language of copulas and describe

Table 2: Standardized VaR of some parametric families and the corresponding worst-case values.

	$\frac{\mathrm{VaR}_\alpha[X]-\mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}}$			Ratio of worst-case value over parametric VaR		
Family / Level α	95%	99%	99.5%	95%	99%	99.5%
Worst-case value (unimodal-symmetric)	2.98	6.67	9.43	-	-	-
Gaussian	1.65	2.33	2.58	1.81	2.86	3.66
Student-T (freedom = 3)	1.36	2.62	3.37	2.20	2.55	2.80
Worst-case value (unimodal)	4.10	9.37	13.30	-	-	-
Exponential (parameter = 1)	2.00	3.61	4.30	2.05	2.60	3.10
Gamma (shape = 0.1, scale = 1)	1.52	4.71	6.31	2.70	2.00	2.11

the tail dependence structure by joint mixability (Wang et al., 2013). First, we list some preliminaries on copulas and joint mixability. Then we provide our main results of sharp RVaR bounds and the corresponding worst scenarios on aggregate risks. Finally, we focus on the worst scenarios of VaR and TVaR of aggregate risks.

3.1 Preliminaries: copulas and joint mixability

A copula is a joint distribution function whose all margins are uniform distributions on $[0,1]$. A copula is used to characterize the dependence structure of a random vector; see Nelsen (2006) for an introduction to copulas. An important copula that we will use later is the comonotonic copula defined as

$$M_n(u_1,\dots,u_n)=\min\{u_1,\dots,u_n\},\quad (u_1,\dots,u_n)\in[0,1]^n.$$

Nelsen (2006, p. 63-64) introduced a method to construct copulas by uniting several different copulas. Here we apply the construction by using two copulas. Suppose $\alpha\in(0,1)$ and C_1,C_2 are two arbitrary n-variate copulas, then for $(u_1,\dots,u_n)\in\mathbb{R}^n$ the function

$$C(u_1,\dots,u_n)=\alpha C_1\left(T^L_\alpha(u_1),\dots,T^L_\alpha(u_n)\right)+(1-\alpha)C_2\left(T^R_\alpha(u_1),\dots,T^R_\alpha(u_n)\right) \tag{4}$$

is also a n-variate copula called an *ordinal sum* of C_1 and C_2, in which

$$T^L_\alpha(x)=\min\left\{\frac{x}{\alpha},1\right\}\quad\text{and}\quad T^R_\alpha(x)=\max\left\{\frac{x-\alpha}{1-\alpha},0\right\},\quad x\in[0,1]. \tag{5}$$

The two functions satisfy that

$$\alpha T_\alpha^L(x) + (1-\alpha) T_\alpha^R(x) = x, \quad x \in [0,1].$$

We write $\mathbf{T}_\alpha^L(u_1, \dots, u_n) = (T_\alpha^L(u_1), \dots, T_\alpha^L(u_n))$ and $\mathbf{T}_\alpha^R = (T_\alpha^R(u_1), \dots, T_\alpha^R(u_n))$.

Remark 4. If a random variable X satisfies $\mathbb{P}(X = \mathrm{VaR}_\alpha[X]) = 0$, then for any $x \in \mathbb{R}$ we have

$$T_\alpha^L(F_X(x)) = \mathbb{P}\left(X \le x \middle| X \le \mathrm{VaR}_\alpha[X]\right) \quad \text{and} \quad T_\alpha^R(F_X(x)) = \mathbb{P}\left(X \le x \middle| X > \mathrm{VaR}_\alpha[X]\right). \tag{6}$$

Moveover, if a random vector $(X_1, \dots, X_n)$ with $\mathbb{P}(X_i = \mathrm{VaR}_\alpha[X_i]) = 0$, $i = 1, \dots, n$ satisfies that its copula can be expressed as (4), then from (6) it can be derived that

$$\mathbb{P}(X_1 > \mathrm{VaR}_\alpha[X_1], \dots, X_n > \mathrm{VaR}_\alpha[X_n] \,|\, X_i > \mathrm{VaR}_\alpha[X_i]) = 1\,, \quad \forall\, i = 1, \dots, n\,,$$

which means that the events that each individual risk exceeds its own VaR at level α occur simultaneously.

The tail part of the sum $X_1 + X_2 + \cdots + X_n$ is essential to the calculation of the worst case of the aggregate risk (Bernard et al., 2014). To describe the dependence structure of the tail part in the worst scenario, we use the concept of joint mixability (Wang et al., 2013), a generalization of complete mixability (Wang and Wang, 2011).

We say n univariate distribution functions $F_1, \dots, F_n$ are *jointly mixable* if there exist n random variables $Z_1, \dots, Z_n$ such that $Z_i \sim F_i$, $i = 1, \dots, n$ and

$$\mathbb{P}(Z_1 + \cdots + Z_n = c) = 1 \quad \text{for some} \ \ c \in \mathbb{R}. \tag{7}$$

It is easy to see that if $Z_i \in L^1$, $i = 1, \dots, n$, then the above constant $c = \sum_{i=1}^n \mathbb{E}[Z_i]$. For any vector $\mathbf{Z} = (Z_1, \dots, Z_n)$ satisfying (7), we call $\mathbf{Z}$ a *joint mix*. For n uniform distributions, joint mixability is equivalent to condition (8) on their lengths in Proposition 1 below. This result is Theorem 3.1 of Wang and Wang (2016), which is key to the existence of the dependence structure of the worst scenarios. This will be used to model the tail dependence structure in later sections.

Proposition 1. *For $t_i > 0$, $i = 1, \dots, n$, the uniform distributions $U(x; 0, t_1), \dots, U(x; 0, t_n)$ are jointly mixable if and only if*

$$\max\{t_1, \dots, t_n\} \le \sum_{i=1}^n t_i/2. \tag{8}$$

Proposition 1 implies that if (8) holds, then there exists a random vector $(Z_1, \ldots, Z_n)$ satisfying $Z_i \sim U(x; 0, t_i),\ i = 1, \ldots, n$ and $\mathbb{P}\left(Z_1 + \cdots + Z_n = \frac{1}{2}(t_1 + \cdots + t_n)\right) = 1$. Denote $F^{JM}_{t_1,\ldots,t_n}$ as the joint distribution function of $(Z_1, \ldots, Z_n)$, then their copula can be represented as

$$C^{JM}_{t_1,\ldots,t_n}(u_1, \ldots, u_n) = F^{JM}_{t_1,\ldots,t_n}(t_1 u_1, \ldots, t_n u_n)\,, \quad (u_1, \ldots, u_n) \in [0,1]^n\,. \tag{9}$$

Throughout the rest of this paper, denote $\mu = \mu_1 + \cdots + \mu_n$, $\sigma = \sigma_1 + \cdots + \sigma_n$, $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and let $M \in \{1, \ldots, n\}$ be such that $\sigma_M = \max\{\sigma_1, \ldots, \sigma_n\}$. Next we define a specific copula family which will be used to describe the tail part of the aggregate risks. Let

$$C^*_{\vec{\sigma}}(u_1, \ldots, u_n) = \begin{cases} C^{JM}_{\sigma_1,\ldots,\sigma_n}(u_1, \ldots, u_n), & \sigma_M \le \sigma/2; \\ \max\left\{0,\ u_k + \min_{i \ne k} u_i - 1\right\}, & \sigma_M = \sigma_k > \sigma/2 \end{cases} \tag{10}$$

for $(u_1, \ldots, u_n) \in [0,1]^n$. One may directly check that (10) defines a copula. Denote

$$\mathcal{C}^{\alpha}_{\vec{\sigma}} = \left\{ \alpha C\left(\mathbf{T}^L_{\alpha}(\cdot)\right) + (1-\alpha) C^*_{\vec{\sigma}}\left(\mathbf{T}^R_{\alpha}(\cdot)\right) \middle|\ C \in \mathcal{C}_n \right\}, \tag{11}$$

where $\mathcal{C}_n$ is the set of n-variate copulas. As we shall see in the following sections, the copula family $\mathcal{C}^{\alpha}_{\vec{\sigma}}$ represents the worst-case dependence scenarios for aggregate risk models.

Remark 5. Copulas in the family $\mathcal{C}^{\alpha}_{\vec{\sigma}}$ is indeed a convex combination of copula $C(u_1, \cdots, u_n)$ and copula $C^*_{\vec{\sigma}}(u_1, \cdots, u_n)$, where $C(u_1, \cdots, u_n)$ is used to model the non-tail part of the aggregate risk and $C^*_{\vec{\sigma}}(u_1, \cdots, u_n)$ is used to model the tail part of the aggregate risk.

3.2 Main results on aggregate risks

Let $X_1, \ldots, X_n$ be individual risks with known means and variances. Additional information such as symmetry and/or unimodality of the individual risks is also considered. For the given partial information, we will discuss the worst-case values of RVaR on the *aggregate risk* $S = X_1 + \cdots + X_n$ when the dependence structure is unspecified. The results on univariate risks in the previous section will be applied.

For the sake of convenience, we define the following two special functions that will be frequently used later. For $\mathcal{V} \in \{V, V_S, V_U, V_{US}\}$ and $0 < \alpha < \beta < 1$, define

$$\mathrm{WR}^{\mathcal{V}(0,1)}(\alpha, \beta; \vec{\sigma}) = \sigma_M \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{\alpha,\beta} + (\sigma - \sigma_M) \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{1+\alpha-\beta,1}, \tag{12}$$

where $\overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{\alpha,\beta}$ and $\overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{1+\alpha-\beta,1}$ are given in Theorem 1, and let

$$\gamma^*_{\mathcal{V}(0,1)} = \arg\min\nolimits_{\gamma\in[\beta,1]} \mathrm{WR}^{\mathcal{V}(0,1)}(\alpha,\gamma;\vec{\sigma}).$$

One can easily check that the above minimum is always attained by some $\gamma^*_{\mathcal{V}(0,1)}$ due to the continuity of $\mathrm{RVaR}_{\alpha,\beta}$ with respect to α and β.

For simplicity, we write for $\mathcal{V} \in \{V, V_S, V_U, V_{US}\}$,

$$\mathcal{V}(\vec{\mu},\vec{\sigma}) = \{X_1 + \cdots + X_n : X_i \in \mathcal{V}(\mu_i,\sigma_i),\ i = 1,\ldots,n\}.$$

The main results, as summarized in the following theorem and Corollary 1 below, give the worst-case values of RVaR and the corresponding worst scenarios.

Theorem 2. *Given $(\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n) \in \mathbb{R}^n \times \mathbb{R}^n_+$ and $\frac{5}{6} \le \alpha < \beta < 1$, we have*

$$\max_{S\in\mathcal{V}(\vec{\mu},\vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] = \mu + \mathrm{WR}^{\mathcal{V}(0,1)}(\alpha,\gamma^*_{\mathcal{V}(0,1)};\vec{\sigma}) = \mu + \min_{\gamma\in[\beta,1]} \mathrm{WR}^{\mathcal{V}(0,1)}(\alpha,\gamma;\vec{\sigma}). \tag{13}$$

Moreover, the upper bound in (13) is attained by $(X_1,\cdots,X_n)$ satisfying the following two conditions:

(1) Marginal distributions: the marginal distributions of $(X_1,\cdots,X_n)$ are given in (b) of Theorem 1 satisfying that $\mathrm{RVaR}_{\alpha,\gamma^*_{\mathcal{V}}(0,1)}[X_M] = \overline{\mathrm{RVaR}}^{\mathcal{V}(\mu_M,\sigma_M)}_{\alpha,\gamma^*_{\mathcal{V}(0,1)}}$ and for $i \neq M$,

$$\mathrm{RVaR}_{1+\alpha-\gamma^*_{\mathcal{V}(0,1)},1}[X_i] = \overline{\mathrm{RVaR}}^{\mathcal{V}(\mu_i,\sigma_i)}_{1+\alpha-\gamma^*_{\mathcal{V}}(0,1),1}.$$

(2) Copula function: the copula C^w of $(X_1,\cdots,X_n)$ can be written as

$$C^w(\cdot) = \begin{cases} M_n(\cdot), & \mathcal{V} = V \text{ or } V_S \\ \alpha C(\mathbf{T}^L_\alpha(\cdot)) + (1-\alpha)C^*_{\vec{\sigma}}(\mathbf{T}^R_\alpha(\cdot))\,, & \mathcal{V} = V_U \text{ or } V_{US}, \end{cases}$$

where C is any copula.

For $i = 1,2,\cdots,n$ the exact distribution of X_i in Theorem 2 can be found in Theorem 1. Theorem 2 states that the copula family $\mathcal{C}^\alpha_{\vec{\sigma}}$ leads to the worst scenarios. As is clarified in Remark 5, the copula $C^*_{\vec{\sigma}}$ is used to model the dependence of the tail part.

(a) In the case $\frac{\sigma_M}{\sigma} \le \frac{1}{2}$, $C^*_{\vec{\sigma}}$ allows n univariate risks with variance $\sigma_1^2,\sigma_2^2,\cdots,\sigma_n^2$ to be joint mixable, which leads to the worst case of the aggregate risks.

(b) In the case $\frac{\sigma_M}{\sigma} > \frac{1}{2}$, $C^*_{\vec{\sigma}}$ in (10) implies that the risk with largest variance is countermonotonic to the other risks that are comonotonic. Precisely, if the copula of $(Z_1, Z_2, \cdots, Z_n)$ can be expressed as

$$C^*_{\vec{\sigma}}(u_1, u_2, \cdots, u_n) = \max\left\{0, u_M + \min_{i \neq M} u_i - 1\right\},$$

then there exists an uniform $[0,1]$ random variable U such that

$$(Z_1, \ldots, Z_n) \stackrel{d}{=} \left(F^{-1}_{Z_1}(U), \ldots, F^{-1}_{Z_{M-1}}(U), F^{-1}_{Z_M}(1-U), F^{-1}_{Z_{M+1}}(U), \ldots, F^{-1}_{Z_n}(U)\right).$$

Remark 6. As was discussed in Wang et al. (2013) and Embrechts et al. (2014), the joint mixability of tail-marginal distributions is the key property to finding worst-case VaR values under dependence uncertainty. It is noted that even when the marginal distributions of $X_1, \ldots, X_n$ are known, finding the worst-case value of $\mathrm{VaR}_p(X_1 + \cdots + X_n)$ is generally an open question. Therefore, Theorem 2 can also be used as an approximation of the worst-case RVaR when the marginal distributions are known (but explicit values of worst-case RVaR are not available), as considered in Embrechts et al. (2014) for worst-case VaR. Another further research direction on this topic is to let n vary; see Embrechts et al. (2015).

The explicit values of $\overline{\mathrm{RVaR}}^{\mathcal{V}}$ determined in Theorem 2 are expressed in the following corollary.

Corollary 1. *Given $(\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \times \mathbb{R}^n_+$, for $\frac{5}{6} \leq \alpha < \beta < 1$, we have*

$$\max_{S \in V(\vec{\mu}, \vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] = \mu + \sigma\sqrt{\tfrac{\alpha}{1-\alpha}},$$

$$\max_{S \in V_S(\vec{\mu}, \vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] = \mu + \sigma\sqrt{\tfrac{1}{2(1-\alpha)}},$$

$$\max_{S \in V_{US}(\vec{\mu}, \vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] = \begin{cases} \mu + \sigma\sqrt{\frac{4}{9(1-\alpha)}}\,, & \frac{\sigma_M}{\sigma} \leq \frac{1}{2}\,; \\ \mu + \sqrt{\frac{1}{2}}\left(\sigma_M^{2/3} + (\sigma - \sigma_M)^{2/3}\right)^{3/2}\sqrt{\frac{4}{9(1-\alpha)}}\,, & \frac{1}{2} < \frac{\sigma_M}{\sigma} \leq \frac{1}{1 + (\frac{\beta-\alpha}{2-\alpha-\beta})^{3/2}}\,; \\ \mu + \sigma_M\sqrt{\frac{4}{9(2-\alpha-\beta)}} + (\sigma - \sigma_M)\sqrt{\frac{4}{9(\beta-\alpha)}}\,, & \frac{\sigma_M}{\sigma} > \frac{1}{1 + (\frac{\beta-\alpha}{2-\alpha-\beta})^{3/2}}\,, \end{cases}$$

and

$$\max_{S \in V_U(\vec{\mu}, \vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] = \begin{cases} \mu + \sigma\sqrt{\frac{8}{9(1-\alpha)} - 1}\,, & \frac{\sigma_M}{\sigma} \leq \frac{1}{2}\,; \\ \mu + \min\limits_{\gamma \in [\beta,1]} (\sigma_M\sqrt{\frac{8}{9(2-\alpha-\gamma)} - 1} + (\sigma - \sigma_M)\sqrt{\frac{8}{9(\gamma-\alpha)} - 1})\,, & \frac{\sigma_M}{\sigma} > \frac{1}{2}\,. \end{cases}$$

3.3 Special cases of VaR and TVaR

From the perspective of capital regulation, VaR and TVaR are the most commonly used risk measures in practice. The sharp bounds on $\mathrm{RVaR}_{\alpha,\beta}[S]$, as obtained in Theorem 2 and Corollary 1, can be applied for VaR and TVaR with the help of Lemma 2.

Before presenting the main conclusions in this subsection, we give some properties of comonotonicity. It is known that if $(X_1,\ldots,X_n)$ is comonotonic, then there exists a $[0,1]$ uniform random variable U such that

$$(X_1,\ldots,X_n) \overset{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\ldots, F_{X_n}^{-1}(U)\right),$$

in which the inverse function $F_{X_i}^{-1}(u) = \inf\{x \in \mathbb{R} \,|\, F_{X_i}(x) > u\}$, $i = 1,\ldots,n$. Moreover, when $n = 2$, we say (X_1, X_2) is countermonotonic if $(X_1, -X_2)$ is comonotonic. If $(X_1,\ldots,X_n)$ is comonotonic, then

$$\mathrm{RVaR}_\alpha[S] = \sum_{i=1}^{n} \mathrm{RVaR}_\alpha[X_i] \quad \forall \alpha \in (0,1), \tag{14}$$

i.e. RVaR is *comonotone additive* (see Kusuoka, 2001).

Since TVaR is subadditive, the worst-case value of aggregate TVaR is the sum of corresponding worst-case values of each individual TVaR. This well-known result can also be obtained by letting $\beta \uparrow 1$ in Theorem 2. The dependence structure which leads to the worst-case value of aggregate TVaR can always be chosen as comonotonicity. We summarize this result in the following corollary.

Corollary 2. *Let $(\vec{\mu}, \vec{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^n_+$. Then*

$$\max_{S \in \mathcal{V}(\vec{\mu},\vec{\sigma})} \mathrm{TVaR}_\alpha[S] = \begin{cases} \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}, & \mathcal{V} = V,\ 0 < \alpha < 1; \\ \mu + \sigma\sqrt{\frac{1}{2(1-\alpha)}}, & \mathcal{V} = V_S,\ \frac{1}{2} < \alpha < 1; \\ \mu + \sigma\sqrt{\frac{8}{9(1-\alpha)} - 1}, & \mathcal{V} = V_U,\ \frac{5}{6} \le \alpha < 1; \\ \mu + \sigma\sqrt{\frac{4}{9(1-\alpha)}}, & \mathcal{V} = V_{US},\ \frac{5}{6} \le \alpha < 1. \end{cases} \tag{15}$$

In the next we will discuss aggregate VaR in each of the four settings. Letting $\beta \downarrow \alpha$ in Theorem 2, by basic calculus it is easy to find $\gamma^*_{\mathcal{V}(0,1)}$ solving the minimum value of the right side of (13) by using the value of $\overline{\mathrm{RVaR}}^{\mathcal{V}}$ derived in Theorem 1. Then we can derive the results on the aggregate VaR.

Corollary 3. *(i) For $\alpha \in (0,1)$,*

$$\max_{S\in V(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] = \mu + \sigma\sqrt{\tfrac{\alpha}{1-\alpha}}; \tag{16}$$

(ii) For $\alpha \in (\frac{1}{2}, 1)$,

$$\max_{S\in V_S(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] = \mu + \sigma\sqrt{\tfrac{1}{2(1-\alpha)}}; \tag{17}$$

(iii) For $\alpha \in [\frac{5}{6}, 1)$,

$$\begin{aligned}&\max_{S\in V_U(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S]\\ &= \begin{cases} \mu + \sigma\sqrt{\frac{8}{9(1-\alpha)} - 1}\,, & \frac{\sigma_M}{\sigma} \le \frac{1}{2}\,;\\ \mu + \min\limits_{\beta\in[\alpha,1]} \left(\sigma_M\sqrt{\frac{8}{9(1-2\alpha+\beta)} - 1} + (\sigma-\sigma_M)\sqrt{\frac{8}{9(1-\beta)} - 1}\right)\,, & \frac{\sigma_M}{\sigma} > \frac{1}{2}\,.\end{cases}\end{aligned} \tag{18}$$

(iv) For $\alpha \in [\frac{5}{6}, 1)$,

$$\max_{S\in V_{US}(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] = \begin{cases} \mu + \sigma\sqrt{\frac{4}{9(1-\alpha)}}\,, & \frac{\sigma_M}{\sigma} \le \frac{1}{2}\,;\\ \mu + \sqrt{\frac{1}{2}}\left(\sigma_M^{2/3} + (\sigma-\sigma_M)^{2/3}\right)^{3/2}\sqrt{\frac{4}{9(1-\alpha)}}\,, & \frac{\sigma_M}{\sigma} > \frac{1}{2}\,.\end{cases} \tag{19}$$

Remark 7. The formula (16) on $S \in V(\vec{\mu},\vec{\sigma})$ was shown in Mesfioui and Quessy (2005, Proposition 4.2), and that part $S \in V_S(\vec{\mu},\vec{\sigma})$ of (17) can be directly shown from Chebyshev's Inequality.

Next we will compare the optimal bounds for VaR and TVaR under different classes. From the above theorem we see that in each of the cases $S \in V(\vec{\mu},\vec{\sigma})$ and $S \in V_S(\vec{\mu},\vec{\sigma})$, the worst-case values of $\mathrm{VaR}_\alpha[S]$ are equal to the corresponding values of $\mathrm{TVaR}_\alpha[S]$, which is consistent with the fact that VaR and TVaR are equal for distribution with two or three point-masses. These results are similar to that of the single risk (see the discussions in Section 2.2).

We analyze the other two cases $S \in V_U(\vec{\mu},\vec{\sigma})$ and $S \in V_{US}(\vec{\mu},\vec{\sigma})$ separately. First, consider the case $S \in V_{US}(\vec{\mu},\vec{\sigma})$. From (15) and (19) we can see that

$$\frac{\max_{S\in V_{US}(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] - \mu}{\max_{S\in V_{US}(\vec{\mu},\vec{\sigma})} \mathrm{TVaR}_\alpha[S] - \mu} = R_{US}\left(\frac{\sigma_M}{\sigma}\right)\,,$$

where the function R_{US} is defined as

$$R_{US}(x) = \begin{cases} 1\,, & x \le \frac{1}{2}\,; \\ \sqrt{\frac{1}{2}}\left(x^{2/3} + (1-x)^{2/3}\right)^{3/2}, & x > \frac{1}{2}\,. \end{cases} \tag{20}$$

This result implies that if $S \in V_{US}(\vec{\mu}, \vec{\sigma})$, the ratio of the standardized worst-case value of aggregate-VaR over aggregate-TVaR is determined by σ_M/σ. Figure 3 shows that the function R_{US} is decreasing, which implies that the worst-case value of aggregate-VaR will be small when there is an individual risk with a relatively large variance. This result shows that the superadditivity of aggregate-VaR is highly relevant to the relative sizes of the individual risks.

For the case $S \in V_U(\vec{\mu}, \vec{\sigma})$, although the bound in (18) for $\sigma_M > \sigma/2$ does not have an explicit expression, we know that the following ratio

$$\frac{\max_{S \in V_U(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] - \mu}{\max_{S \in V_U(\vec{\mu},\vec{\sigma})} \mathrm{TVaR}_\alpha[S] - \mu}$$

is determined by σ_M/σ and α, and hence we denote it by $R_U(\sigma_M/\sigma, \alpha)$.

Moreover, by numerical calculation we find that $R_U(x, \alpha) \approx R_{US}(x)$ when α is close to 1. Figure 3 shows the effect of this approximation when $\alpha = 0.95$. Based on the above analysis, (19) can be rewritten as

$$\max_{S \in V_{US}(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S] \approx \mu + R_{US}\left(\frac{\sigma_M}{\sigma}\right) \cdot \sigma \sqrt{\frac{8}{9(1-\alpha)} - 1}\,.$$

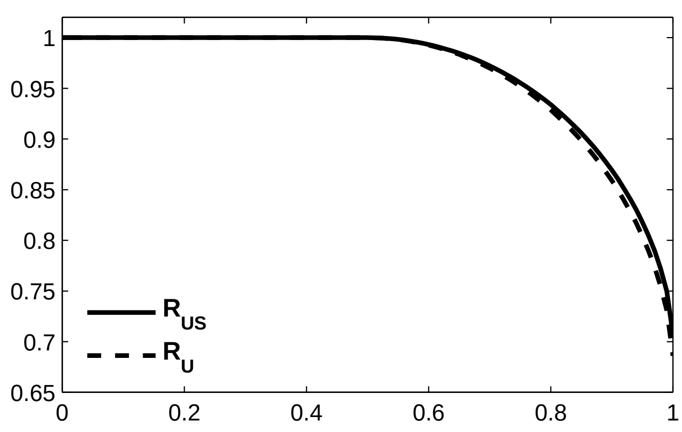

Figure 3: The dotted line is the function $R_{US}(x)$, $x \in [0,1]$ defined in (20); the dashed line is $R_U(x, 0.95)$, $x \in [0,1]$.

To summarize, one can see that the values $\max_{S \in \mathcal{V}(\vec{\mu},\vec{\sigma})} \mathrm{VaR}_\alpha[S]$ and $\max_{S \in \mathcal{V}(\vec{\mu},\vec{\sigma})} \mathrm{TVaR}_\alpha[S]$

are quite close for many choices of $\mathcal{V}$, α, $\vec{\mu}$ and $\vec{\sigma}$. This phenomenon is similar to the VaR/ES asymptotic equivalence under dependence uncertainty; see Embrechts et al. (2015).

4 Proofs of main results

4.1 Proof of Theorem 1

In this subsection, we first recall the definition and some lemmas on convex order, then we give the proof of Theorem 1.

A random variable X is said to be smaller than another random variable Y in *convex order*, written as $X \le_{cx} Y$, if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for any convex function $\phi : \mathbb{R} \mapsto \mathbb{R}$ provided that the expectations exist.

Lemma 3. *(Shaked and Shanthikumar, 2007, p.109–120)*

(i) For $X \in L^1$, $\mathbb{E}[X] \le_{cx} X$;

(ii) Let $X, Y \in L^2$. If $Y \le_{cx} X$, then $\mathbb{E}[X] = \mathbb{E}[Y]$, $\mathrm{Var}(Y) \le \mathrm{Var}(X)$, $-Y \le_{cx} -X$ and $\mathrm{RVaR}_{0,p}[Y] \ge \mathrm{RVaR}_{0,p}[X]$ for any $p \in (0,1)$;

(iii) Suppose $\alpha_i \ge 0$, $i = 1, \dots, n$ with $\alpha_1 + \cdots + \alpha_n = 1$ and $Y_i \le_{cx} X_i$ for $i = 1, \dots, n$. If $X \sim \sum_{i=1}^n \alpha_i F_{X_i}$ and $Y \sim \sum_{i=1}^n \alpha_i F_{Y_i}$, then $Y \le_{cx} X$.

The following lemma gives a convenient criterion for convex order.

Lemma 4. *(Shaked and Shanthikumar, 2007, p.133) For $X, Y \in L^1$ with $\mathbb{E}[X] = \mathbb{E}[Y]$, if F_X up-crosses F_Y, i.e. there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} F_X(x) \le F_Y(x), & x < x_0; \\ F_X(x) \ge F_Y(x), & x > x_0, \end{cases}$$

then $X \le_{cx} Y$.

Applying the above criterion, we have the following lemma.

Lemma 5. *Supposing that X is a nonnegative continuous random variable with mean $\mu > 0$, and its density function f_X is decreasing on $[0, \infty)$. Let $Y \sim U(x; 0, 2\mu)$, then $Y \le_{cx} X$ and $f_X(0) \ge 1/(2\mu)$.*

Proof. The proof of $Y \leq_{cx} X$ can be found in Theorem 3.A.46 of Shaked and Shanthikumar (2007). Moreover, if $f_X(0) < 1/(2\mu)$, then $f_X(x) < 1/(2\mu)$ for any $x \in [0, 2\mu]$, so

$$\begin{aligned}\mathbb{E}X - \mu &= \int_{2\mu}^{\infty} x f_X(x)dx - \int_0^{2\mu} x\left(\frac{1}{2\mu} - f_X(x)\right)dx \\ &\geq 2\mu\int_{2\mu}^{\infty} f_X(x)dx - 2\mu\int_0^{2\mu}\left(\frac{1}{2\mu} - f_X(x)\right)dx = 0,\end{aligned}$$

which is contradictory to $\mathbb{E}X = \mu$. Hence we conclude that $f_X(0) \geq 1/(2\mu)$. □

To begin with the proof of Theorem 1, we introduce some notation. Denote

$$\begin{aligned}
D_\gamma(x;a,b) &= \gamma I[x \geq a] + (1-\gamma)I[x \geq b], & \mathcal{D} &= \{Y : Y \sim D_\gamma(x;a,b),\ a \leq b,\ \gamma \in [0,1]\};\\
D_\gamma^M(x;a,b) &= \gamma D_{\frac{1}{2}}(x,a,b) + (1-\gamma)I[x \geq \tfrac{a+b}{2}], & \mathcal{D}_M &= \left\{Y : Y \sim D_\gamma^M(x;a,b),\ a \leq b,\ \gamma \in [0,1]\right\};\\
U_\gamma^L(x;a,b) &= \gamma U(x;a,b) + (1-\gamma)I[x \geq a], & \mathcal{U}_L &= \{Y : Y \sim U_\gamma^L(x;a,b),\ a \leq b,\ \gamma \in [0,1]\};\\
U_\gamma^R(x;a,b) &= \gamma U(x;a,b) + (1-\gamma)I[x \geq b], & \mathcal{U}_R &= \{Y : Y \sim U_\gamma^R(x;a,b),\ a \leq b,\ \gamma \in [0,1]\};\\
U_\gamma^M(x;a,b) &= \gamma U(x;a,b) + (1-\gamma)I[x \geq \tfrac{a+b}{2}], & \mathcal{U}_M &= \{Y : Y \sim U_\gamma^M(x;a,b),\ a \leq b,\ \gamma \in [0,1]\}.
\end{aligned}$$

Hence $D_\gamma(x;a,b)$ and $D_\gamma^M(x;a,b)$ are discrete distributions, and $\mathcal{D}$ and $\mathcal{D}_M$ are the families of the corresponding discrete random variables. Moreover, $U_\gamma^L(x;a,b)$, $U_\gamma^R(x;a,b)$ and $U_\gamma^M(x;a,b)$ respectively are mixture distributions of uniform distribution and a point mass, and $\mathcal{U}_L$,$\mathcal{U}_R$ and $\mathcal{U}_M$ denote the families of the corresponding random variables.

It is easy to verify that if $Y \sim G_\theta$ (resp. $G_\theta^S, G_\theta^U, G_\theta^{US}$), then $Y \in \mathcal{D}$ (resp. $\mathcal{D}_M$,$\mathcal{U}_L$,$\mathcal{U}_M$). We will prove in Proposition 2 below that the random variable in $V(\mu,\sigma)$ (resp. $V_S(\mu,\sigma)$, $V_U(\mu,\sigma)$, $V_{US}(\mu,\sigma)$) with the largest value of RVaR belongs to the family $\mathcal{D}$ (resp. $\mathcal{D}_M$, $\mathcal{U}_L \cup \mathcal{U}_R$, $\mathcal{U}_M$). First we need the following lemma.

Lemma 6. *Let $a \in \mathbb{R}$ and X be a continuous unimodal random variable with support $[a,\infty)$. If $Y \sim U_\gamma^R(x;\, a,b)$ for some $b > a$, $\gamma \in [0,1]$ satisfies $\mathbb{E}[Y] = \mathbb{E}[X]$ and $f_X(a) \geq \frac{\gamma}{b-a}$, then $Y \leq_{cx} X$.*

Proof. Let m denote the mode of X, then $m \geq a$. The following proof is divided into the two cases $m \geq b$ and $m < b$.

Case 1: $m \geq b$. Since X is unimodal, $f_X(x) \geq f_X(a) \geq \frac{\gamma}{b-a}$ for $x \in [a,b)$. Then

$$F_X(x) = \int_a^x f_X(s)ds \geq \frac{\gamma(x-a)}{b-a} = U_\gamma^R(x;\, a,b), \quad x \in [a,b).$$

On the other hand, $F_X(x) \le 1 = U^R_\gamma(x;\, a, b)$ for $x \in (b, \infty)$. Therefore $U^R_\gamma(x;\, a, b)$ up-crosses F_X at b, and from Lemma 4 we conclude $Y \le_{cx} X$.

Case 2: $m < b$. Similarly, we have $F_X(x) \ge \frac{\gamma(x-a)}{b-a} = U^R_\gamma(x;\, a, b)$ for $x \in [a, m]$ and $F_X(x) \le 1 = U^R_\gamma(x;\, a, b)$ for $x \in [b, \infty)$. Note that by the unimodality of X we know that $F_X(x)$ is a concave function on $[m, b]$. Together with the fact that $U^R_\gamma(x;\, a, b)$ is a convex function on $[m, b]$, we know that $U^R_\gamma(x;\, a, b)$ up-crosses $F_X(x)$ at some point in $[m, b]$, and hence by Lemma 4 we have $Y \le_{cx} X$. $\square$

Given $\alpha \in (0, 1)$ and a random variable X, we define two random variables X^L_α, X^R_α as

$$X^L_\alpha \sim T^L_\alpha(F_X(x)) \quad \text{and} \quad X^R_\alpha \sim T^R_\alpha(F_X(x)), \tag{21}$$

where the functions

$$T^L_\alpha(x) = \min\left\{\frac{x}{\alpha}, 1\right\} \quad \text{and} \quad T^R_\alpha(x) = \max\left\{\frac{x-\alpha}{1-\alpha}, 0\right\}, \quad x \in [0, 1]$$

are introduced in (5). It is easy to verify

$$F_X(x) = \alpha F_{X^L_\alpha}(x) + (1-\alpha) F_{X^R_\alpha}(x), \tag{22}$$

and for any $0 < \alpha < \beta < 1$,

$$\mathrm{VaR}_\beta[X] = \mathrm{VaR}_{(\beta-\alpha)/(1-\alpha)}[X^R_\alpha]. \tag{23}$$

Particularly, if $X \in L^1$ and $\mathbb{P}(X = \mathrm{VaR}_\alpha[X]) = 0$, from (6) we know that

$$F_{X^L_\alpha}(x) = \mathbb{P}(X \le x |\, X < \mathrm{VaR}_\alpha[X]) \quad \text{and} \quad F_{X^R_\alpha}(x) = \mathbb{P}(X \le x |\, X > \mathrm{VaR}_\alpha[X]).$$

Moreover, for any $\alpha \in (\frac{1}{2}, 1)$ we let

$$X^M_\alpha \sim F_{X^M_\alpha}(x) = \frac{1}{2\alpha - 1}\left(F_X(x) - (1-\alpha)T^L_{1-\alpha}(F_X(x)) - (1-\alpha)T^R_\alpha(F_X(x))\right). \tag{24}$$

It is easy to check that the function in the right side above is a distribution function. Note that if X is symmetric, then X^M_α is also symmetric. Particularly, if $X \in L^1$ and $\mathbb{P}(X = \mathrm{VaR}_\alpha[X]) = 0$, then

$$F_{X^M_\alpha}(x) = \mathbb{P}(X \le x |\, \mathrm{VaR}_{1-\alpha}[X] \le X \le \mathrm{VaR}_\alpha[X]).$$

The following lemma establishes the relationship between convex order and RVaR.

Lemma 7. *(i) Given $\alpha \in (0,1)$, if $X, Y \in L^2$ satisfy that $Y_\alpha^L \le_{cx} X_\alpha^L$ and $Y_\alpha^R \le_{cx} X_\alpha^R$, then* $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

(ii) Given $\alpha \in (\frac{1}{2}, 1)$, if $X, Y \in L^2$ satisfy that $Y_{1-\alpha}^L \le_{cx} X_{1-\alpha}^L$, $Y_\alpha^M \le_{cx} X_\alpha^M$ and $Y_\alpha^R \le_{cx} X_\alpha^R$, then $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

Proof. (i) By the definition of X_α^L, X_α^R in (21) and equation (22), applying Lemma 3(iii) we conclude that $Y \le_{cx} X$ according to $Y_\alpha^L \le_{cx} X_\alpha^L$ and $Y_\alpha^R \le_{cx} X_\alpha^R$. Hence $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ follows.

On the other hand, $Y_\alpha^R \le_{cx} X_\alpha^R$ implies that $\mathrm{RVaR}_{0,p}[Y_\alpha^R] \ge \mathrm{RVaR}_{0,p}[X_\alpha^R]$ holds for any $p \in (0,1)$ by applying Lemma 3(ii). Together with (23), we have that for any $\beta \in (\alpha, 1)$,

$$\mathrm{RVaR}_{\alpha,\beta}[Y] = \mathrm{RVaR}_{0,(\beta-\alpha)/(1-\alpha)}[Y_\alpha^R] \ge \mathrm{RVaR}_{0,(\beta-\alpha)/(1-\alpha)}[X_\alpha^R] = \mathrm{RVaR}_{\alpha,\beta}[X]. \tag{25}$$

(ii) From (24) we have

$$F_X(x) = (1-\alpha)F_{X_{1-\alpha}^L}(x) + (2\alpha - 1)F_{X_\alpha^M}(x) + (1-\alpha)F_{X_\alpha^R}(x).$$

Similarly, we can conclude $Y \le_{cx} X$ by Lemma 3(iii). Hence $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ follows. Moreover, by the same argument in (25) we also have $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ for $\beta \in (\alpha, 1)$ in this case.

□

Proposition 2. *(i) For any random variable $X \in L^2$ and $\alpha \in (0,1)$, there exists $Y \in \mathcal{D}$ such that* $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

(ii) For any symmetric random variable $X \in L^2$ and $\alpha \in (\frac{1}{2}, 1)$, there exists $Y \in \mathcal{D}_M$ such that $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

(iii) For any continuous unimodal random variable $X \in L^2$ and $\alpha \in (0,1)$, there exists $Y \in \mathcal{U}_L \cup \mathcal{U}_R$ such that $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

(iv) For any continuous unimodal-symmetric random variable $X \in L^2$ and $\alpha \in (\frac{1}{2}, 1)$, there exists $Y \in \mathcal{U}_M$ such that $\mathrm{Var}(Y) \le \mathrm{Var}(X)$ *and* $\mathrm{RVaR}_{\alpha,\beta}[Y] \ge \mathrm{RVaR}_{\alpha,\beta}[X]$ *for any $\beta \in (\alpha, 1)$.*

Proof. (i) Recall X_α^L and X_α^R defined in (21). Let $Y \sim D_\alpha(x; \mathbb{E}[X_\alpha^L], \mathbb{E}[X_\alpha^R])$, then $Y \in \mathcal{D}$. From the facts $Y_\alpha^L = \mathbb{E}[X_\alpha^L] \le_{cx} X_\alpha^L$ and $Y_\alpha^R = \mathbb{E}[X_\alpha^R] \le_{cx} X_\alpha^R$, we get the conclusions by Lemma 7.

(ii) Let $Y \sim D_{2-2\alpha}(x; \mathbb{E}[X_{1-\alpha}^L], \mathbb{E}[X_\alpha^R])$, then $Y \in \mathcal{D}_M$. Notice that $Y_{1-\alpha}^L = \mathbb{E}[X_{1-\alpha}^L] \le_{cx} X_{1-\alpha}^L$, $Y_\alpha^R = \mathbb{E}[X_\alpha^R] \le_{cx} X_\alpha^R$ and $Y_\alpha^M = \mu = \mathbb{E}[X_\alpha^M] \le_{cx} X_\alpha^M$ owing to the symmetry of X. Then applying Lemma 7, we can get the conclusion.

(iii) We first prove the following inequality

$$f_X\left(\mathrm{VaR}_\alpha[X]\right) \geq \min\{g_L, g_R\}, \tag{26}$$

where

$$g_L = \frac{\alpha}{2}(\mathrm{VaR}_\alpha[X] - \mathbb{E}[X_\alpha^L])^{-1} \quad \text{and} \quad g_R = \frac{1-\alpha}{2}(\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X])^{-1}.$$

Denote by m the mode of X. If $m \leq \mathrm{VaR}_\alpha[X]$, then $Z = X_\alpha^R - \mathrm{VaR}_\alpha[X]$ is a nonnegative continuous random variable with decreasing density over $[0, \infty)$. Applying Lemma 5 we get

$$(2\mathbb{E}[X_\alpha^R] - 2\mathrm{VaR}_\alpha[X])^{-1} \leq f_Z(0) = f_X(\mathrm{VaR}_\alpha[X])/(1-\alpha),$$

hence $f_X\left(\mathrm{VaR}_\alpha[X]\right) \geq g_R$ follows. Otherwise, in the case $m > \mathrm{VaR}_\alpha[X]$, $\mathrm{VaR}_\alpha[X] - X_\alpha^L$ is a nonnegative continuous random variable with decreasing density over $[0, \infty)$. Similarly, we can derive $f_X(\mathrm{VaR}_\alpha[X]) \geq g_L$ by applying Lemma 5. Thus combining the above two cases, we get (26).

Next we set

$$Y \sim F_Y(x) = \begin{cases} U_{d_1}^L\left(x;\, \mathrm{VaR}_\alpha[X] - \frac{1-\sqrt{1-g_R/g_L}}{g_R/\alpha},\, 2\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X]\right), & g_R \leq g_L; \\ U_{d_2}^R\left(x;\, 2\mathbb{E}[X_\alpha^L] - \mathrm{VaR}_\alpha[X],\, \mathrm{VaR}_\alpha[X] + \frac{1-\sqrt{1-g_L/g_R}}{g_L/(1-\alpha)}\right), & g_R > g_L, \end{cases} \tag{27}$$

where

$$d_1 = (1-\alpha) + \alpha(1 - \sqrt{1 - g_R/g_L})$$

if $g_R \leq g_L$, and

$$d_2 = \alpha + (1-\alpha)(1 - \sqrt{1 - g_L/g_R})$$

if $g_R > g_L$. Then $Y \in \mathcal{U}_L \cup \mathcal{U}_R$. If $Y_\alpha^L \leq_{cx} X_\alpha^L$ and $Y_\alpha^R \leq_{cx} X_\alpha^R$, then we can get the conclusion by applying Lemma 7. Thus in the next, we will show $Y_\alpha^L \leq_{cx} X_\alpha^L$ and $Y_\alpha^R \leq_{cx} X_\alpha^R$ by considering the two cases $g_R \leq g_L$ and $g_R > g_L$ respectively.

Case 1: $g_R \leq g_L$. According to (27), we can check that

$$F_{Y_\alpha^R}(x) = U\left(x;\, \mathrm{VaR}_\alpha[X],\, 2\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X]\right),$$

hence $\mathbb{E}[Y_\alpha^R] = \mathbb{E}[X_\alpha^R]$ follows. Note that X_α^R is a continuous unimodal random variable on

$[\mathrm{VaR}_\alpha[X], \infty)$. Furthermore, from (26) we can derive that the density of X_α^R satisfies that

$$f_{X_\alpha^R}\left(\mathrm{VaR}_\alpha[X]\right) = \frac{1}{1-\alpha} f_X\left(\mathrm{VaR}_\alpha[X]\right) \geq \frac{g_R}{1-\alpha} = \frac{1}{2(\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X])}.$$

Then applying Lemma 6 we conclude $Y_\alpha^R \leq_{cx} X_\alpha^R$. On the other hand, we can also check

$$F_{Y_\alpha^L}(x) = U_{d_1^*}^L\left(x;\, \mathrm{VaR}_\alpha[X] - \tfrac{1-\sqrt{1-g_R/g_L}}{g_R/\alpha},\, \mathrm{VaR}_\alpha[X]\right) \quad \text{with} \quad d_1^* = 1 - \sqrt{1 - g_R/g_L}$$

and $\mathbb{E}[Y_\alpha^L] = \mathbb{E}[X_\alpha^L]$. Note that X_α^L is a continuous unimodal random variable on $(-\infty, \mathrm{VaR}_\alpha[X]]$ and its density at point $\mathrm{VaR}_\alpha[X]$ satisfies

$$f_{X_\alpha^L}\left(\mathrm{VaR}_\alpha[X]\right) = \frac{1}{\alpha} f_X\left(\mathrm{VaR}_\alpha[X]\right) \geq \frac{g_R}{\alpha}.$$

Thus applying Lemma 6 we have $-Y_\alpha^L \leq_{cx} -X_\alpha^L$, hence $Y_\alpha^L \leq_{cx} X_\alpha^L$ follows.

Case 2: $g_R > g_L$. Similarly, according to (27) we can check that

$$F_{X_\alpha^R}(x) = U_{d_2^*}^R\left(x;\, \mathrm{VaR}_\alpha[X],\, \mathrm{VaR}_\alpha[X] + \frac{1-\sqrt{1-g_L/g_R}}{g_L/(1-\alpha)}\right) \quad \text{with} \quad d_2^* = 1 - \sqrt{1 - g_L/g_R}$$

and $\mathbb{E}[Y_\alpha^R] = \mathbb{E}[X_\alpha^R]$. Since X_α^R is a continuous unimodal random variable on $[\mathrm{VaR}_\alpha[X], \infty)$ with $f_{X_\alpha^R}\left(\mathrm{VaR}_\alpha[X]\right) \geq \frac{g_R}{1-\alpha}$, then we can derive $Y_\alpha^R \leq_{cx} X_\alpha^R$ by Lemma 6.

On the other hand, we can also check

$$F_{X_\alpha^L}(x) = U\left(x;\, 2\mathbb{E}[X_\alpha^L] - \mathrm{VaR}_\alpha[X],\, \mathrm{VaR}_\alpha[X]\right) = U_1^L\left(x;\, 2\mathbb{E}[X_\alpha^L] - \mathrm{VaR}_\alpha[X],\, \mathrm{VaR}_\alpha[X]\right),$$

and hence $\mathbb{E}[Y_\alpha^L] = \mathbb{E}[X_\alpha^L]$. Since X_L is a continuous unimodal random variable on $(-\infty, \mathrm{VaR}_\alpha[X]]$ with $f_{X_L}\left(\mathrm{VaR}_\alpha[X]\right) \geq \frac{g_L}{\alpha}$, we can derive $-Y_\alpha^L \leq_{cx} -X_\alpha^L$ by Lemma 6, hence $Y_\alpha^L \leq_{cx} X_\alpha^L$ follows.

(iv) By the same argument as in the proof of (26), we note that $Z = X_\alpha^R - \mathrm{VaR}_\alpha[X]$ is a nonnegative continuous random variable with decreasing density over $[0, \infty)$. Then applying Lemma 5 we have

$$f_X(\mathrm{VaR}_\alpha[X]) = (1-\alpha) f_Z(0) \geq \frac{1-\alpha}{2(\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X])} = g_R.$$

Combining with that X is unimodal and symmetric, we get

$$f_X(x) \geq f_X\left(\mathrm{VaR}_\alpha[X]\right) \geq g_R\,, \quad \forall\ \mathrm{VaR}_{1-\alpha}[X] \leq x \leq \mathrm{VaR}_\alpha[X]\,. \tag{28}$$

Denote $d_3^* = 2g_R(\mathrm{VaR}_\alpha[X] - \mathbb{E}[X])/(2\alpha - 1)$. Note that

$$2\alpha - 1 = F_X(\mathrm{VaR}_\alpha[X]) - F_X(\mathrm{VaR}_{1-\alpha}[X]) = \int_{\mathrm{VaR}_{1-\alpha}[X]}^{\mathrm{VaR}_\alpha[X]} f_X(x)dx \geq 2g_R(\mathrm{VaR}_\alpha[X] - \mathbb{E}[X]),$$

in which the last inequality is due to the symmetry of X and (28). Therefore, $d_3^* \in [0,1]$.

Set $d_3 = 2(1-\alpha) + (2\alpha - 1)d_3^* \in [0,1]$ and

$$Y \sim F_Y(x) = U_{d_3}^M\left(x;\, 2\mathbb{E}[X_{1-\alpha}^L] - \mathrm{VaR}_{1-\alpha}[X],\, 2\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X]\right)\,,$$

then $Y \in \mathcal{U}_M$. If $Y_{1-\alpha}^L \leq_{cx} X_{1-\alpha}^L$, $Y_\alpha^M \leq_{cx} X_\alpha^M$ and $Y_\alpha^R \leq_{cx} X_\alpha^R$, we obtain the conclusion by Lemma 7. Thus in the next we show $Y_{1-\alpha}^L \leq_{cx} X_{1-\alpha}^L$, $Y_\alpha^M \leq_{cx} X_\alpha^M$ and $Y_\alpha^R \leq_{cx} X_\alpha^R$ separately.

First we show $Y_\alpha^M \leq_{cx} X_\alpha^M$. It is easy to check $Y_\alpha^M \sim U_{d_3^*}^M\left(x;\, \mathrm{VaR}_{1-\alpha}[X],\, \mathrm{VaR}_\alpha[X]\right)$. Moreover,

$$F_{X_\alpha^M}(x) = \int_{\mathrm{VaR}_{1-\alpha}[X]}^{x} \frac{f_X(y)}{2\alpha - 1}dy \geq \frac{g_R(x - \mathrm{VaR}_{1-\alpha}[X])}{(2\alpha - 1)} = F_{Y_\alpha^M}(x), \quad x \in \left(\mathrm{VaR}_{1-\alpha}[X],\, \mathbb{E}[X]\right),$$

where the second inequality is due to (28). By the same argument, we have $F_{X_\alpha^M}(x) \leq F_{Y_\alpha^M}(x)$ for $x \in (\mathbb{E}[X],\, \mathrm{VaR}_\alpha[X])$. Therefore $F_{Y_\alpha^M}$ up-crosses $F_{X_\alpha^M}$ once at $\mathbb{E}[X]$, and hence $Y_\alpha^M \leq_{cx} X_\alpha^M$ by Lemma 4.

Next we show $Y_\alpha^R \leq_{cx} X_\alpha^R$ and $Y_{1-\alpha}^L \leq_{cx} X_{1-\alpha}^L$. We can also check $Y_\alpha^R \sim U(x;\, \mathrm{VaR}_\alpha[X],\, 2\mathbb{E}[X_\alpha^R] - \mathrm{VaR}_\alpha[X])$. Thus by Lemma 5 we know $Y_\alpha^R \leq_{cx} X_\alpha^R$. Owing to the symmetry of X, we also know that $Y_{1-\alpha}^L \leq_{cx} X_{1-\alpha}^L$. Thus the proof is finished. □

Finally, we arrive at a proof of Theorem 1.

Proof of Theorem 1. (i) Consider the case $X \in V(\mu, \sigma)$. We claim that for any $0 < \alpha < \beta < 1$,

$$\max_{X \in V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] = \max_{Y \in \mathcal{D} \cap V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[Y]. \tag{29}$$

Note that $\mathcal{D} \cap V(\mu,\sigma) \subsetneqq V(\mu,\sigma)$. Thus in order to prove (29), it is sufficient to prove that for any $X^* \in V(\mu,\sigma)$, there exists $Y \in \mathcal{D} \cap V(\mu,\sigma)$ such that $\mathrm{RVaR}_{\alpha,\beta}[Y] \geq \mathrm{RVaR}_{\alpha,\beta}[X^*]$. If

$\mathrm{RVaR}_{\alpha,\beta}[X^*] \le \mu$, we can choose $Y \sim G_\alpha(\frac{x-\mu}{\sigma})$, then $Y \in \mathcal{D} \cap V(\mu,\sigma)$ and

$$\mathrm{RVaR}_{\alpha,\beta}[Y] = \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}} \ge \mathrm{RVaR}_{\alpha,\beta}[X^*]$$

follows. Else if $\mathrm{RVaR}_{\alpha,\beta}[X^*] > \mu$, by Proposition 2 there exists some $Z \in \mathcal{D}$ such that $\mathrm{Var}(Z) \le \sigma^2$ and $\mathrm{RVaR}_{\alpha,\beta}[Z] \ge \mathrm{RVaR}_{\alpha,\beta}[X^*]$, and for $Y = \mu + \sigma(Z-\mu)/\sqrt{\mathrm{Var}(Z)} \in \mathcal{D} \cap V(\mu,\sigma)$ we can calculate

$$\mathrm{RVaR}_{\alpha,\beta}[Y] = \mathrm{RVaR}_{\alpha,\beta}[Z] + \left(\frac{\sigma}{\sqrt{\mathrm{Var}(Z)}} - 1\right)(\mathrm{RVaR}_{\alpha,\beta}[Z] - \mu) \ge \mathrm{RVaR}_{\alpha,\beta}[X^*].$$

Thus (29) holds.

Finally we can calculate

$$\max_{Y \in \mathcal{D} \cap V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[Y] = \mu + \sigma\sqrt{\tfrac{\alpha}{1-\alpha}}, \quad \forall\, 0 < \alpha < \beta < 1,$$

and the equality holds when $Y \sim G_\alpha\left(\frac{x-\mu}{\sigma}\right)$.

(ii) Consider the case $X \in V_S(\mu,\sigma)$. Similarly, we have for any $\frac{1}{2} \le \alpha < \beta < 1$,

$$\max_{X \in V_S(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] = \max_{Y \in \mathcal{D}_M \cap V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[Y] = \mu + \sigma\sqrt{\tfrac{1}{2(1-\alpha)}},$$

in which the second equality holds when $X \sim G^S_\alpha\left(\frac{x-\mu}{\sigma}\right)$.

(iii-iv) Consider the cases $X \in V_U(\mu,\sigma)$ and $X \in V_{US}(\mu,\sigma)$. Since each unimodal distribution can be approximated by a sequence of continuous unimodal distributions, here without loss of generality we assume that X is continuous and unimodal distributed. By the same argument as in part (i), we have that for any $\frac{5}{6} \le \alpha < \beta < 1$,

$$\max_{X \in V_U(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] = \max_{Y \in (\mathcal{U}_L \cup \mathcal{U}_R) \cap V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[Y] = \mu + \sigma\sqrt{\tfrac{8}{9(2-\alpha-\beta)} - 1},$$

and

$$\max_{X \in V_{US}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] = \max_{Y \in \mathcal{U}_M \cap V(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[Y] = \mu + \sigma\sqrt{\tfrac{4}{9(2-\alpha-\beta)}},$$

where the equalities hold when $X \sim G^U_{\alpha+\beta-1}\left(\frac{x-\mu}{\sigma}\right)$ and $X \sim G^{US}_{\alpha+\beta-1}\left(\frac{x-\mu}{\sigma}\right)$ respectively. □

4.2 Proof of Lemma 2

By (3) in Theorem 1, the value of $\mathrm{RVaR}_{\alpha,\beta}^{\mathcal{V}(\mu,\sigma)}$ is continuous in β for $\beta > \alpha$, therefore we can explicitly calculate the limit

$$\lim_{\beta\downarrow\alpha} \sup_{X\in\mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] = \begin{cases} \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\,, & \mathcal{V} = V,\ 0 < \alpha < 1; \\ \mu + \sigma\sqrt{\frac{1}{2(1-\alpha)}}\,, & \mathcal{V} = V_S,\ \frac{1}{2} < \alpha < 1; \\ \mu + \sigma\sqrt{\frac{4}{9(1-\alpha)} - 1}\,, & \mathcal{V} = V_U,\ \frac{5}{6} \le \alpha < 1; \\ \mu + \sigma\sqrt{\frac{2}{9(1-\alpha)}}\,, & \mathcal{V} = V_{US},\ \frac{5}{6} \le \alpha < 1, \end{cases} \tag{30}$$

which is equal to the right-hand side of (3) by setting $\beta = \alpha$. Since $\mathrm{RVaR}_{\alpha,\beta}[X] \ge \mathrm{VaR}_\alpha[X]$ for $\beta > \alpha$, we have

$$\lim_{\beta\downarrow\alpha} \sup_{X\in\mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X] \ge \sup_{X\in\mathcal{V}(\mu,\sigma)} \mathrm{VaR}_\alpha[X]. \tag{31}$$

To show the converse direction of (31), take X^* as a random variable with distribution $G_\alpha(\frac{x-\mu}{\sigma})$, $G^S_\alpha(\frac{x-\mu}{\sigma})$, $G^U_{2\alpha-1}(\frac{x-\mu}{\sigma})$, or $G^{US}_{2\alpha-1}(\frac{x-\mu}{\sigma})$ under each of the four settings of $\mathcal{V}$ as V, V_S, V_U, or V_{US} respectively. One can directly calculate that $\mathrm{VaR}_\alpha[X^*]$ is also equal to the right-hand side of (30) as reported in Table 1. Therefore,

$$\sup_{X\in\mathcal{V}(\mu,\sigma)} \mathrm{VaR}_\alpha[X] \ge \mathrm{VaR}_\alpha[X^*] = \lim_{\beta\downarrow\alpha} \sup_{X\in\mathcal{V}(\mu,\sigma)} \mathrm{RVaR}_{\alpha,\beta}[X],$$

and together with (31) we obtain equality.

The proof of the TVaR equality in this lemma is obtained by a similar argument.

4.3 Proof of Theorem 2

We first give an inequality on RVaR for the sum of two risks and then generalize it to the sum of n risks.

Lemma 8. *For two random variables X_1, X_2 and $0 < \alpha < \beta \le 1$, the following inequality holds*

$$\mathrm{RVaR}_{\alpha,\beta}[X_1 + X_2] \le \mathrm{RVaR}_{\alpha,\beta}[X_1] + \mathrm{TVaR}_{1+\alpha-\beta}[X_2].$$

Proof. Let $V \sim U[0,1]$ such that $X_1 = F_{X_1}^{-1}(V)$. For some $m < \mathrm{VaR}_\alpha(X_1)$, let

$$Y = X_1 I_{\{V \le \beta\}} + m I_{\{V > \beta\}}.$$

We have $\mathrm{TVaR}_{1-\beta+\alpha}[Y] = \mathrm{RVaR}_{\alpha,\beta}[X_1]$.

Note that for $t \in \mathbb{R}$, $\{X_1 + X_2 > t\} \subset \{Y + X_2 > t\} \cup \{Y \neq X_1\}$, and hence

$$\mathbb{P}(X_1 + X_2 > t) \le \mathbb{P}(Y + X_2 > t) + \mathbb{P}(Y \neq X_1) \le \mathbb{P}(Y + X_2 > t) + (1 - \beta).$$

It follows that

$$\mathbb{P}(Y + X_2 \le t) \le \mathbb{P}(X_1 + X_2 \le t) + (1 - \beta). \tag{32}$$

By the definition of VaR, (32) implies that for $p \in (0, 1-\beta)$,

$$\mathrm{VaR}_p[X_1 + X_2] \le \mathrm{VaR}_{p+1-\beta}[Y + X_2]. \tag{33}$$

Therefore, we have

$$\begin{aligned}
\mathrm{RVaR}_{\alpha,\beta}[X_1 + X_2] &= \frac{1}{\beta - \alpha} \int_\alpha^\beta \mathrm{VaR}_p[X_1 + X_2] dp \\
&\le \frac{1}{\beta - \alpha} \int_\alpha^\beta \mathrm{VaR}_{p+1-\beta}[Y + X_2] dp \\
&= \mathrm{TVaR}_{1-\beta+\alpha}[Y + X_2] \\
&\le \mathrm{TVaR}_{1-\beta+\alpha}[Y] + \mathrm{TVaR}_{1-\beta+\alpha}[X_2] \\
&= \mathrm{RVaR}_{\alpha,\beta}(X_1) + \mathrm{TVaR}_{1-\beta+\alpha}[X_2],
\end{aligned}$$

where the first inequality is due to (33) and the second inequality is due to the subadditivity of TVaR. □

Proposition 3. *Let* $0 < \alpha < \beta \le \gamma \le 1$. *Then*

$$\mathrm{RVaR}_{\alpha,\beta}[S] \le \mathrm{RVaR}_{\alpha,\gamma}[X_1] + \sum_{i=2}^n \mathrm{TVaR}_{1+\alpha-\gamma}[X_i]\,. \tag{34}$$

Proof. This proposition follows immediately from $\mathrm{RVaR}_{\alpha,\beta}[S] \le \mathrm{RVaR}_{\alpha,\gamma}[S]$ and Lemma 8. □

Remark 8. Letting $\beta \downarrow \alpha$ and $\gamma = 1$ in Proposition 3, the inequality is simplified as the well-known inequality

$$\mathrm{VaR}_\alpha[S] \le \sum_{i=1}^n \mathrm{TVaR}_\alpha[X_i]\,. \tag{35}$$

It is obvious that inequality (34) is stronger than (35).

We are ready to give a proof of Theorem 2.

Proof of Theorem 2. We first show the upper bound in the right side of (13), then we show that the upper bound is attainable. Without loss of generality we assume $\sigma_1 = \max\{\sigma_1, \dots, \sigma_n\}$.

By Proposition 3, we have that for every $\gamma \in [\beta, 1)$ and $X_i \in \mathcal{V}(\mu_i, \sigma_i)$, $i = 1, \dots, n$,

$$\begin{aligned}\mathrm{RVaR}_{\alpha,\beta}[S] \le& \mathrm{RVaR}_{\alpha,\gamma}[X_1] + \sum_{i=2}^n \mathrm{TVaR}_{1+\alpha-\gamma}[X_i] \\ \le& \mu + \sigma_1 \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{\alpha,\gamma} + \sum_{i=2}^n \sigma_i \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{1+\alpha-\gamma,1}.\end{aligned}$$

Thus

$$\begin{aligned}\max_{S\in\mathcal{V}(\vec{\mu},\vec{\sigma})} \mathrm{RVaR}_{\alpha,\beta}[S] &\le \mu + \min_{\gamma\in[\beta,1]}\left\{\sigma_1 \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{\alpha,\gamma} + \sum_{i=2}^n \sigma_i \cdot \overline{\mathrm{RVaR}}^{\mathcal{V}(0,1)}_{1+\alpha-\gamma,1}\right\} \\ &= \mu + \mathrm{WR}^{\mathcal{V}(0,1)}(\alpha, \gamma^*_{\mathcal{V}(0,1)}; \vec{\sigma})\end{aligned} \tag{36}$$

follows. In the next we will see that the bound in (36) is attainable for each choice of $\mathcal{V}$.

(a). Consider $S \in V(\vec{\mu},\vec{\sigma})$ and $S \in V_S(\vec{\mu},\vec{\sigma})$. In the case $S \in V(\vec{\mu},\vec{\sigma})$, it is easy to verify that $\gamma^*_{\mathcal{V}(0,1)} = 1$ gives the minimum of (36). Hence we set $X_i \sim G_\alpha(\frac{x-\mu}{\sigma})$, $i = 1, \dots, n$ and let $(X_1, \dots, X_n)$ be comonotonic, then from (14) and Theorem 1 we know the bounds are attained. A similar argument can be used for the case $S \in V_S(\vec{\mu},\vec{\sigma})$.

(b). Consider $S \in V_U(\vec{\mu},\vec{\sigma})$. We discuss the two separate cases for $S \in V_U(\vec{\mu},\vec{\sigma})$. For simplicity, we write $\gamma^*_{V_U(0,1)}$ as γ^*.

Case 1: $\sigma_M \le \sigma/2$. From simple analysis, $\gamma^* = 1$ solves the minimum of the right side of (13). Combined with $\overline{\mathrm{RVaR}}^{\mathcal{V}}_{\alpha,1}$ in Theorem 1, we know the upper bound

$$\mu + \mathrm{WR}^{\mathcal{V}(0,1)}(\alpha, \gamma^*; \vec{\sigma}) = \mu + \sigma\sqrt{\frac{8}{9(1-\alpha)} - 1}.$$

Take a uniform random variable W on $[0,1]$ and

$$\mathbf{Y}=(Y_1,\ldots,Y_n)\sim C\left(T_\alpha^L\left(G_\alpha^U\left(\frac{x-\mu_1}{\sigma_1}\right)\right),\ldots,T_\alpha^L\left(G_\alpha^U\left(\frac{x-\mu_n}{\sigma_n}\right)\right)\right),$$
$$\mathbf{Z}=(Z_1,\ldots,Z_n)\sim C^{JM}_{\sigma_1,\ldots,\sigma_n}\left(T_\alpha^R\left(G_\alpha^U\left(\frac{x-\mu_1}{\sigma_1}\right)\right),\ldots,T_\alpha^R\left(G_\alpha^U\left(\frac{x-\mu_n}{\sigma_n}\right)\right)\right),$$

and let $W,\mathbf{Y},\mathbf{Z}$ be independent. Denote

$$X_i=Y_i\cdot I[W\le\alpha]+Z_i\cdot I[W>\alpha],i=1,\ldots,n.$$

Then it is obvious to see that $X_i\sim G_\alpha^U(\frac{x-\mu_i}{\sigma_i})$, $i=1,\ldots,n$ and their copula belongs to $\mathcal{C}^\alpha_{\vec{\sigma}}$. Note that

$$S=I[W\le\alpha]\sum_{i=1}^n Y_i+I[W>\alpha]\sum_{i=1}^n Z_i.$$

Next we prove that $\mathrm{RVaR}_{\alpha,\beta}[S]$ attains its upper bound in this case. From the definition of G_α^U, we can check

$$T_\alpha^R\left(G_\alpha^U\left(\tfrac{x-\mu_i}{\sigma_i}\right)\right)=U(x;\ \mu_i+\sigma_iB_1,\ \mu_i+\sigma_iB_2) \tag{37}$$

in which the constants $B_1=\left(1-\frac{4}{9\alpha-1}\right)\sqrt{\frac{8}{9(1-\alpha)}-1}$ and $B_2=\left(1+\frac{4}{9\alpha-1}\right)\sqrt{\frac{8}{9(1-\alpha)}-1}$. Therefore, we know $(Z_i-\mu_i)/\sigma_i\sim U(x;B_1,B_2)$, $i=1,\ldots,n$. From the definition of $C^{JM}_{\sigma_1,\ldots,\sigma_n}$, we know

$$Z_1+\cdots+Z_n=\mu+\sigma\cdot(B_1+B_2)/2=\mu+\sigma\sqrt{\frac{8}{9(1-\alpha)}-1}.$$

Since $Y_i\le Z_i$, $i=1,\ldots,n$ according to the definition of T_α^L,T_α^R in (6), we have

$$\mathrm{RVaR}_{\alpha,\beta}[S]=Z_1+\cdots+Z_n=\mu+\sigma\sqrt{\frac{8}{9(1-\alpha)}-1}=\mu+\mathrm{WR}^{\mathcal{V}(0,1)}(\alpha,\gamma^*;\vec{\sigma}).$$

Case 2: $\sigma_M>\sigma/2$. Define $\mathbf{Y}=(Y_1,Y_2,\cdots,Y_n)$ with distribution

$$C\left(T_\alpha^L(G^U_{\alpha+\gamma^*-1}(\frac{x-\mu_1}{\sigma_1})),T_\alpha^L(G^U_{1+\alpha-\gamma^*}(\frac{x-\mu_2}{\sigma_2})),\ldots,T_\alpha^L(G^U_{1+\alpha-\gamma^*}(\frac{x-\mu_n}{\sigma_n}))\right)$$

and $\mathbf{Z}=(Z_1,Z_2,\cdots,Z_n)$ with distribution

$$\max\left\{0,\ T_\alpha^R\left(G^U_{\alpha+\gamma^*-1}\left(\frac{x-\mu_1}{\sigma_1}\right)\right)+\min_{2\le i\le n}T_\alpha^R\left(G^U_{1+\alpha-\gamma^*}\left(\frac{x-\mu_i}{\sigma_i}\right)\right)-1\right\}.$$

Let W be a $[0,1]$ uniform random variable, and $W, \mathbf{Y}, \mathbf{Z}$ be independent and

$$X_i = Y_i \cdot I_{\{W \le \alpha\}} + Z_i \cdot I_{\{W > \alpha\}}, i = 1, \ldots, n.$$

Then

$$S = I_{\{W \le \alpha\}} \sum_{i=1}^{n} Y_i + I_{\{W > \alpha\}} \sum_{i=1}^{n} Z_i.$$

Next we will analyze $Z_1 + \cdots + Z_n$. Note that the copula of $Z_1, \ldots, Z_n$ is $\max\{0,\ u_1 + \min_{2 \le i \le n} u_i - 1\}$, thus $Z_2, \ldots, Z_n$ are comonotonic and Z_1, Z_2 are countermonotonic. Then $Z_1 + \cdots + Z_n$ has identical distribution with $\mathrm{VaR}_V[Z_1] + \mathrm{VaR}_{1-V}[Z_2] + \cdots + \mathrm{VaR}_{1-V}[Z_n]$, in which V is a $[0,1]$ uniform random variable. By definition and simple analysis, $\mathrm{VaR}_V[Z_1 + \cdots + Z_n]$ has the same distribution with $\mathrm{VaR}_V[Z_1] + \mathrm{VaR}_{1-V}[Z_2] + \cdots + \mathrm{VaR}_{1-V}[Z_n]$. In the next we focus on $\gamma^* = \beta$ and $\gamma^* > \beta$ separately.
(i). If $\gamma^* = \beta$, from above analysis we have

$$\begin{aligned} \mathrm{RVaR}_{\alpha,\beta}[S] &= \mathrm{RVaR}_{0,\frac{\beta-\alpha}{1-\alpha}}[Z_1 + Z_2 + \cdots + Z_n] \\ &= \mathrm{RVaR}_{0,\frac{\beta-\alpha}{1-\alpha}}[Z_1] + \sum_{i=2}^{n} \mathrm{TVaR}_{\frac{1-\beta}{1-\alpha}}[Z_i] \\ &= \mathrm{RVaR}_{\alpha,\beta}[X_1] + \sum_{i=2}^{n} \mathrm{TVaR}_{1+\alpha-\beta}[X_i], \end{aligned}$$

then the proof completes.
(ii). If $\gamma^* > \beta$, then $\gamma^* = \arg\min_{\gamma \in (\beta, 1]} \mathrm{WR}^{V_U(0,1)}(\alpha, \beta; \vec{\sigma})$, in which

$$\mathrm{WR}^{V_U(0,1)}(\alpha, \beta; \vec{\sigma}) = \sigma_1 \sqrt{\frac{8}{9(2-\alpha-\beta)} - 1} + (\sigma - \sigma_1)\sqrt{\frac{8}{9(\beta-\alpha)} - 1}.$$

By simple calculation we have

$$\frac{d}{d\beta} \mathrm{WR}^{V_U(0,1)}(\alpha, \beta; \vec{\sigma}) \bigg|_{\beta=1} > 0.$$

Hence from the definition of γ^* and the assumption that $\gamma^* > \beta$, we have

$$\frac{d}{d\beta} \mathrm{WR}^{V_U(0,1)}(\alpha, \beta; \vec{\sigma}) \mid_{\beta=\gamma^*} = \frac{4}{3} \left(\frac{\sigma_1}{(2-\alpha-\gamma^*)^2 \sqrt{-9 + \frac{8}{2-\alpha-\gamma^*}}} - \frac{\sigma - \sigma_1}{(\alpha-\gamma^*)^2 \sqrt{-9 + \frac{8}{\gamma^*-\alpha}}} \right) = 0.$$

Thus we get

$$\frac{\sigma_1}{(9(\alpha+\gamma^*)+8)(2-\alpha-\gamma^*)}\sqrt{\frac{8}{9(2-\alpha-\gamma^*)}-1}=\frac{\sigma-\sigma_1}{(9(\alpha-\gamma^*)+8)(\gamma^*-\alpha)}\sqrt{\frac{8}{9(\gamma^*-\alpha)}-1}. \tag{38}$$

Note that $X_1 \sim G^U_{\alpha+\gamma^*-1}$ and $X_i \sim G^U_{1+\alpha-\gamma^*}$ for $i=2,\dots,n$. For $\xi \sim G^U_\theta(x)$ with $\theta \in [\frac{2}{3},1)$,

$$\mathrm{VaR}_u[\xi]=\begin{cases}\left(1+\frac{8}{9\theta-1}\cdot\frac{u-(\theta+1)/2}{1-\theta}\right)\sqrt{\frac{8}{9(1-\theta)}-1}, & u\geq\frac{3\theta-1}{2};\\ -\sqrt{\frac{9(1-\theta)}{9\theta-1}}, & u<\frac{3\theta-1}{2},\end{cases}$$

thus

$$\mathrm{VaR}_u[\xi]\geq\left(1+\frac{8}{9\theta-1}\cdot\frac{u-(\theta+1)/2}{1-\theta}\right)\sqrt{\frac{8}{9(1-\theta)}-1}, \quad \forall\, u\in(0,1). \tag{39}$$

As a consequence of (39), we have for any $p\in[0,1]$,

$$\begin{aligned}\mathrm{VaR}_p[Z_1]&=\mathrm{VaR}_{\alpha+p(1-\alpha)}[X_1]\\ &\geq\mu_1+\sigma_1\left(1+\frac{8}{9(\alpha+\gamma^*-1)-1}\cdot\frac{\alpha+p(1-\alpha)-(\alpha+\gamma^*)/2}{2-\alpha-\gamma^*}\right)\sqrt{\frac{8}{9(2-\alpha-\gamma^*)}-1},\end{aligned} \tag{40}$$

and for $i=2,\dots,n$,

$$\begin{aligned}\mathrm{VaR}_{1-p}[Z_i]&=\mathrm{VaR}_{\alpha+(1-p)(1-\alpha)}[X_i]\\ &\geq\mu_i+\sigma_i\left(1+\frac{8}{9(\alpha-\gamma^*)+8}\cdot\frac{\alpha+(1-p)(1-\alpha)-(2+\alpha-\gamma^*)/2}{\gamma^*-\alpha}\right)\sqrt{\frac{8}{9(\gamma^*-\alpha)}-1}.\end{aligned} \tag{41}$$

Integrating (40) and (41), together with (38), we have for any $p\in[0,1]$,

$$\begin{aligned}&\mathrm{VaR}_p[Z_1]+\sum_{i=2}^n\mathrm{VaR}_{1-p}[Z_i]\\ &\geq\mu+\mathrm{WR}^{V_U(0,1)}(\alpha,\gamma^*;\vec{\sigma})+8(\alpha+p(1-\alpha-(\alpha+\gamma^*)/2))\times\\ &\quad\left(\frac{\sigma_1}{(9(\alpha+\gamma^*)+8)(2-\alpha-\gamma^*)}\sqrt{\frac{8}{9(2-\alpha-\gamma^*)}-1}-\frac{\sigma-\sigma_1}{(9(\alpha-\gamma^*)+8)(\gamma^*-\alpha)}\sqrt{\frac{8}{9(\gamma^*-\alpha)}-1}\right)\\ &=\mu+\mathrm{WR}^{V_U(0,1)}(\alpha,\gamma^*;\vec{\sigma}).\end{aligned} \tag{42}$$

Since (42) holds for any $p\in[0,1]$, we obtain

$$Z_1+\dots+Z_n\geq\mu+\mathrm{WR}^{V_U(0,1)}(\alpha,\gamma^*;\vec{\sigma}).$$

Applying the property of T^L_α, T^R_α in (6), we know $Y_i \le Z_i$, $i = 1, \dots, n$, hence $\mathrm{RVaR}_{\alpha,\beta}[S] \ge \mu + \mathrm{WR}^{V_U(0,1)}(\alpha, \gamma^*; \vec{\sigma})$. On the other hand, $\mu + \mathrm{WR}^{V_U(0,1)}(\alpha, \gamma^*; \vec{\sigma})$ is the right side of (13). Hence proof completes.

(c) For the case $S \in V_{US}(\vec{\mu}, \vec{\sigma})$, the proof is similar to that of the case $S \in V_U(\vec{\mu}, \vec{\sigma})$.

$\square$

5 Conclusion

In this paper, we study worst-case values of the risk measure RVaR in the presence of model uncertainty with known mean and variance of the risks in individual and aggregate models. Descriptive information such as symmetry and unimodality of the univariate risks is incorporated in the analysis. Analytical formulas for sharp worst-case values, and their corresponding marginal distributions and dependence structures are obtained based on stochastic comparison and joint mixability. The two most relevant limit cases of RVaR, VaR and TVaR, are analyzed in details. The results in this paper can be used to deliver conservative capital requirement calculation and provide an analytical reference for stress testing.

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